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I.—MEMOIR OF THE LATE D. F. GREGORY, M.A., FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

By R. Leslie Ellis, Esq., Fellow of Trinity College, Cambridge.

The subject of the following memoir died in his thirty-first year. He had, nevertheless, accomplished enough not only to justify high expectations of his future progress in the science to which he had principally devoted himself, but also to entitle his name to a place in some permanent record.

Duncan Farquharson Gregory was born at Edinburgh in April 1813. He was the youngest son of Dr. James Gregory, the distinguished professor of Medicine, and was thus of the same family as the two celebrated mathematicians James and David Gregory. The former of these, his direct ancestor, is familiarly remembered as the inventor of the telescope which bears his name; he lived in an age of great mathematicians, and was not unworthy to be their contemporary.

Of the early years of Mr. Gregory's life but little need be said. The peculiar bent of his mind towards mathematical speculations does not appear to have been perceived during his childhood; but, in the usual course of education, he shewed much facility in the acquisition of knowledge, a remarkably active and inquiring mind, and a very retentive memory. It may, perhaps, be mentioned here, that his father, whom he lost before he was seven years old, used to predict distinction for him; and was so struck with his accurate information and clear memory, that he had pleasure in conversing with him, as with an equal, on subjects of history and geography. In his case, as in many others, ingenuity in little mechanical contrivances seems to have preceded, and indicated the developement of a taste for abstract science.

Two years of his life were passed at the Edinburgh Academy; when he left it, being considered too young for the University, he went abroad and spent a winter at a pri-

vate academy in Geneva. Here his talent for mathematics attracted attention; in geometry, as well as in classical learning, he had already made distinguished progress at Edinburgh.

The following winter he attended classes at the University of Edinburgh, and soon became a favourite pupil of Professor Wallace's, under whose tuition he made great advances in the higher parts of mathematics. The Professor formed the highest hopes of Mr. Gregory's future eminence: those who long afterwards saw them together in Cambridge, speak with much interest of the delighted pride he shewed in his pupil's success and increasing reputation.

In 1833, Mr. Gregory's name was entered at Trinity College in the University of Cambridge, and shortly afterwards he went to reside there. He brought with him a very unusual amount of knowledge on almost all scientific subjects: with Chemistry he was particularly well acquainted, so much so that he had been at Cambridge but a few months when it was proposed to him by one of the most distinguished men in the University to act as assistant to the professor of Chemistry; which for some time he did. Indeed, it is impossible to doubt that, had not other pursuits engaged his attention, he might have achieved a great reputation as a chemist. He was one of the founders of the Chemical Society in Cambridge, and occasionally gave lectures in their rooms.

He had also a very considerable knowledge of botany, and indeed of many subjects which he seemed never to have studied systematically: he possessed in a remarkable degree the power of giving a regular form, and, so to speak, a unity to knowledge acquired in fragments.

All these tastes and habits of thought Mr. Gregory cultivated, to a certain extent, during the first years of his residence in Cambridge, of course in subordination to that which was the end principally in view in his becoming a member of the University, namely, the study of mathematics and natural philosophy.

He became a Bachelor of Arts in 1837, having taken high mathematical honours: more, however, might, we may believe, have been effected in this respect, had his activity of mind permitted him to devote himself more exclusively to the prescribed course of study.

From henceforth he felt himself more at liberty to follow original speculations, and, not many months after taking his degree, turned his attention to the general theory of the combination of symbols.

It may be well to say a few words of the history of this part of mathematics.

One of the first results of the differential notation of Leibnitz, was the recognition of the analogy of differentials and powers. For instance, it was readily perceived that

$$\frac{d^{m+n}}{dx^{m+n}}\, y = \frac{d^m}{dx^m}\,\frac{d^n}{dx^n}\, y,$$

or, supposing the y to be *understood*, that

$$\left(\frac{d}{dx}\right)^{m+n} = \left(\frac{d}{dx}\right)^m \left(\frac{d}{dx}\right)^n,$$

just as in ordinary algebra we have, a being any quantity,

$$a^{m+n} = a^m a^n.$$

This, and one or two other remarks of the same kind, were sufficient to establish an analogy between $\frac{d}{dx}$ the symbol of differentiation and the ordinary symbols of algebra. And it was not long afterwards remarked that a corresponding analogy existed between the latter class of symbols and that which is peculiar to the calculus of finite differences. It was inferred from hence that theorems proved to be true of combinations of ordinary symbols of quantity, might be applied by analogy to the differential calculus and to that of finite differences. The meaning and interpretation of such theorems would of course be wholly changed by this kind of transfer from one part of mathematics to another, but their form would remain unchanged. By these considerations many theorems were suggested, of which it was thought almost impossible to obtain direct demonstrations. In this point of view the subject was developed by Lagrange, who left undemonstrated the results to which he was led, intimating, however, that demonstrations were required. Gradually, however, mathematicians came to perceive that the analogy with which they were dealing, involved an essential identity; and thus results, with respect to which, if the expression may be used, it had only been felt that they must be true, were now actually seen to be so. For, if the algebraical theorems by which these results were suggested, were true, *because* the symbols they involve represented quantities, and such operations as may be performed on quantities, then indeed the analogy would be altogether precarious. But if, as is really the case, these theorems are true, in virtue of certain fundamental laws of combination, which hold both for algebraical

symbols, and for those peculiar to the higher branches of mathematics, then each algebraical theorem and its analogue constitute, in fact, only one and the same theorem, except *quoad* their distinctive interpretations, and therefore a demonstration of either is in reality a demonstration of both.*

The abstract character of these considerations is doubtless the reason why so long a time elapsed before their truth was distinctly perceived. They would almost seem to require, in order that they may be readily apprehended, a peculiar faculty—a kind of mental *disinvoltura* which is by no means common.

Mr. Gregory, however, possessed it in a very remarkable degree. He at once perceived the truth and the importance of the principles of which we have been speaking, and proceeded to apply them with singular facility and fearlessness.

It had occurred to two or three distinguished writers that the analogy, as it was called, of powers, differentials, &c., might be made available in the solution of differential equations, and of equations in finite differences.

This idea, however, probably from some degree of doubt as to the legitimacy of the methods which it suggested, had not been fully or clearly developed: it seems to have been chiefly employed as affording a convenient way of expressing solutions already obtained by more familiar considerations.

To this branch of the subject Mr. Gregory directed his attention, and from the general views of the laws of combination of symbols already noticed, deduced in a regular and systematic form, methods of solution of a large and important class of differential equations (linear equations with constant coefficients, whether ordinary or partial) of systems of such equations existing simultaneously, of the corresponding classes of equations in finite and mixed differences; and lastly, of many functional equations. The steady and unwavering apprehension of the fundamental principle which pervades all these applications of it, gives them a value quite independent of that which arises from the facility of the methods of solution which they suggest.

The investigations of which I have endeavoured to illustrate the character and tendency, appeared from time to time in the *Cambridge Mathematical Journal.*

* The values of certain definite integrals are to be looked upon as merely arithmetical results; in such cases we are not at liberty to replace the constants involved in the definite integrals by symbols of operation. In other cases we are at liberty to do so, and this remarkable application of the principles stated in the text, has already led Mr. Boole of Lincoln, with whom it seems to have originated, to several curious conclusions.

In this periodical publication Mr. Gregory took much interest. He had been active in establishing it, and continued to be its editor, except for a short interval, from the time of its first appearance in the autumn of 1837, until a few months before his death. For this occupation he was for many reasons well qualified; his acquaintance with mathematical literature was very extensive, while his interest in all subjects connected with it was not only very strong, but also singularly free from the least tinge of jealous or personal feeling. That which another had done or was about to do, seemed to give him as much pleasure as if he himself had been the author of it, and this even when it related to some subject which his own researches might seem to have appropriated.

This trait, as the recollections of those who knew him best will bear me witness, was intimately connected with his whole character, which was in truth an illustration of the remark of a French writer, that to be free from envy is the surest indication of a fine nature.

To the *Cambridge Mathematical Journal*, Mr. Gregory contributed many papers beside those which relate to the researches already noticed. In some of these he developed certain particular applications of the principles he had laid down in an Essay on the Foundations of Algebra, presented to the Royal Society of Edinburgh in 1838, and printed in the fourteenth volume of their Transactions. I may particularly mention a paper on the curious question of the logarithms of negative quantities, a question which, it is well known, has often been discussed among mathematicians, and which even now does not appear to be entirely settled.

In 1840, Mr. Gregory was elected Fellow of Trinity College; in the following year he became Master of Arts, and was appointed to the office of moderator, that is, of principal mathematical examiner. His discharge of the duties of this office (which is looked upon as one of the most honourable of those which are accessible to the younger members of the University) was distinguished by great good sense and discretion.

In the close of the year 1841, Mr. Gregory produced his "Collection of Examples of the Processes of the Differential and Integral Calculus;" a work which required, and which manifests much research, and an extensive acquaintance with mathematical writings. He had at first only wished to superintend the publication of a second edition of the work with a similar title, which appeared more than twenty-five years

since, and of which Messrs. Herschel, Peacock, and Babbage were the authors. Difficulties, however, arose, which prevented the fulfilment of this wish, and it is not perhaps to be regretted that Mr. Gregory was thus led to undertake a more original design. It is well known that the earlier work exercised a great and beneficial influence on the studies of the University, nor was it in any way unworthy of the reputation of its authors. The original matter contributed by Sir John Herschel is especially valuable. Nevertheless, the progress which mathematical science has since made, rendered it desirable that another work of the same kind should be produced, in which the more recent improvements of the calculus might be embodied.

Since the beginning of the century, the general aspect of mathematics has greatly changed. A different class of problems from that which chiefly engaged the attention of the great writers of the last age has arisen, and the new requirements of natural philosophy have greatly influenced the progress of pure analysis. The mathematical theories of heat, light, electricity, and magnetism, may be fairly regarded as the achievement of the last fifty years. And in this class of researches an idea is prominent, which comparatively occurs but seldom in purely dynamical enquiries. This is the idea of discontinuity. Thus, for instance, in the theory of heat, the conditions relating to the surface of the body whose variations of temperature we are considering, form an essential and peculiar element of the problem; their peculiarity arises from the discontinuity of the transition from the temperature of the body to that of the space in which it is placed. Similarly, in the undulatory theory of light, there is much difficulty in determining the conditions which belong to the bounding surfaces of any portion of ether; and although this difficulty has, in the ordinary applications of the theory, been avoided by the introduction of proximate principles, it cannot be said to have been got rid of.

The power, therefore, of symbolizing discontinuity, if such an expression may be permitted, is essential to the progress of the more recent applications of mathematics to natural philosophy, and it is well known that this power is intimately connected with the theory of definite integrals. Hence the principal importance of this theory, which was altogether passed over in the earlier collection of examples.

Mr. Gregory devoted to it a chapter of his work, and noticed particularly some of the more remarkable applications of definite integrals to the expression of the solutions of

partial differential equations. It is not improbable that in another edition he would have developed this subject at somewhat greater length. He had long been an admirer of Fourier's great work on heat, to which this part of mathematics owes so much; and once, while turning over its pages, remarked to the writer,—"All these things seem to me to be a kind of mathematical paradise."

In 1841, the mathematical Professorship at Toronto was offered to Mr. Gregory: this, however, circumstances induced him to decline. Some years previously he had been a candidate for the Mathematical Chair at Edinburgh.

His year of office as moderator ended in October 1842. In the University Examination for Mathematical Honours in the following January, he, however, in accordance with the usual routine, took a share, with the title of examiner,—a position little less important, and very nearly as laborious, as that of moderator. Besides these engagements in the University, he had been for two or three years actively employed in lecturing and examining in the College of which he was a Fellow. In the fulfilment of these duties, he shewed an earnest and constant desire for the improvement of his pupils, and his own love of science tended to diffuse a taste for it among the better order of students. He had for some time meditated a work on Finite Differences, and had commenced a treatise on Solid Geometry, which, unhappily, he did not live to complete. In the midst of these various occupations, he felt the earliest approaches of the malady which terminated his life.

The first attack of illness occurred towards the close of 1842. It was succeeded by others, and in the spring of 1843, he left Cambridge never to return again. He had just before taken part in a college examination, and notwithstanding severe suffering, had gone through the irksome labour of examining with patient energy and undiminished interest.

Many months followed of almost constant pain. Whenever an interval of tolerable ease occurred, he continued to interest himself in the pursuits to which he had been so long devoted; he went on with the work on Geometry, and, but a little while before his death, commenced a paper on the analogy of differential equations and those in finite differences. This analogy it is known that he had developed to a great length; unfortunately, only a portion of his views on the subject can now be ascertained.

At length, on the 23d February 1844, after sufferings, on

which, notwithstanding the admirable patience with which they were borne, it would be painful to dwell, his illness terminated in death. He had been for a short time aware that the end was at hand, and, with an unclouded mind, he prepared himself calmly and humbly for the great change; receiving and giving comfort and support from the thankful hope that the close of his suffering life here, was to be the beginning of an endless existence of rest and happiness in another world. He retained to the last, when he knew that his own connection with earthly things was soon to cease, the unselfish interest which he had ever felt in the pursuits and happiness of those he loved.

A few words may be allowed about a character where rare and sterling qualities were combined. His upright, sincere, and honourable nature secured to him general respect. By his intimate friends, he was admired for the extent and variety of his information, always communicated readily, but without a thought of display,—for his refinement and delicacy of taste and feeling,—for his conversational powers and playful wit; and he was beloved by them for his generous, amiable disposition, his active and disinterested kindness, and steady affection. And in this manner his high-toned character acquired a moral influence over his contemporaries and juniors, in a degree remarkable in one so early removed.

To this brief history, little more is to be added; for though it is impossible not to indulge in speculations as to all that Mr. Gregory might have done in the cause of science and for his own reputation, had his life been prolonged, yet such speculations are necessarily too vague to find a place here; and even were it not so, it would perhaps be unwise to enter on a subject so full of sources of unavailing regret.

II.—ON THE PARTIAL DIFFERENTIAL EQUATIONS TO A FAMILY OF ENVELOPS.

By W. Walton, M.A. Trinity College.

The subject of this paper is the following general problem: "To investigate the partial differential equation of the envelop of a surface, the equation to which involves three variable parameters, the parameters themselves being subject to two unknown equations of relation."

The solution of problems of this class has not yet, as far as I am aware, been attempted by symmetrical methods: my object is to supply this deficiency. I shall begin with the

particular cases of Developable and Tubular surfaces: I shall then proceed to the consideration of the problem in all its generality. The arguments which I have laid down in the discussion of the two individual problems, and in the general one, are so precisely similar as to appear tautological: I have chosen, however, so to express myself, in order that each of the divisions of the paper may be separately intelligible.

In the following researches I shall put, for the sake of brevity,

$$\frac{du}{dx} = a, \quad \frac{du}{dy} = b, \quad \frac{du}{dz} = c,$$

$$\frac{d^2u}{dx^2} = a', \quad \frac{d^2u}{dy^2} = b', \quad \frac{d^2u}{dz^2} = c',$$

$$\frac{d^2u}{dydz} = a'', \quad \frac{d^2u}{dzdx} = b'', \quad \frac{d^2u}{dxdy} = c''.$$

1. Let x, y, z, be the co-ordinates of any point of a developable surface; α, β, γ, the variable parameters. Then

$$\alpha x + \beta y + \gamma z = 1 \ldots\ldots\ldots\ldots\ldots (1),$$

$$x d\alpha + y d\beta + z d\gamma = 0 \ldots\ldots\ldots\ldots\ldots (2);$$

α, β, γ, being subject to two equations

$$F(\alpha, \beta, \gamma) = 0, \quad f(\alpha, \beta, \gamma) = 0 \ldots\ldots\ldots (3).$$

From (1) and (2) there is

$$\alpha dx + \beta dy + \gamma dz = 0 \ldots\ldots\ldots\ldots (4).$$

Suppose $u = 0$ to be the equation to the developable surface: then we shall have also

$$adx + bdy + cdz = 0 \ldots\ldots\ldots\ldots (5).$$

By the aid of an indeterminate multiplier λ we shall get from (4) and (5), observing that by virtue of (1), (2), (3), x and y may be regarded as independent variables,

$$\alpha = \frac{a}{\lambda}, \quad \beta = \frac{b}{\lambda}, \quad \gamma = \frac{c}{\lambda} \ldots\ldots\ldots (6).$$

Now the only equations connecting $\alpha, \beta, \gamma, x, y, z$, with $d\alpha, d\beta, d\gamma$, are (2) and the differentials of (3); all which three equations are satisfied identically by putting

$$d\alpha = 0, \quad d\beta = 0, \quad d\gamma = 0,$$

without subjecting to any limitation the absolute or relative values of $x, y, z, \alpha, \beta, \gamma$. Differentiating, then, equations (6) on this hypothesis, we get

$$\frac{d\lambda}{\lambda} = \frac{da}{a} = \frac{1}{a} \cdot (a'dx + c''dy + b''dz),$$

$$\frac{d\lambda}{\lambda} = \frac{db}{b} = \frac{1}{b} \cdot (b'dy + a''dz + c''dx),$$

$$\frac{d\lambda}{\lambda} = \frac{dc}{c} = \frac{1}{c} \cdot (c'dz + b''dx + a''dy).$$

Eliminating dy and dz by cross-multiplication, we get

$$abc \cdot \frac{d\lambda}{\lambda} \cdot \{a(b'c' - a''^2) + b(a''b'' - c'c'') + c(c''a'' - b'b'')\} = Vdx \ldots (7),$$

where $\qquad V = a'b'c' - a'a''^2 - b'b''^2 - c'c''^2 + 2a''b''c''.$

Observing that V is a symmetrical function of a', b', c', a'', b'', c'', it is evident that we shall have also

$$abc \cdot \frac{d\lambda}{\lambda} \cdot \{b(c'a' - b''^2) + c(b''c'' - a'a'') + a(a''b'' - c'c'')\} = Vdy \ldots (8),$$

$$abc \cdot \frac{d\lambda}{\lambda} \cdot \{c(a'b' - c''^2) + a(c''a'' - b'b'') + b(b''c'' - a'a'')\} = Vdz \ldots (9).$$

Multiplying these equations (7), (8), (9), by a, b, c, respectively, adding, and attending to (5), we get

$$a^2(b'c' - a''^2) + b^2(c'a' - b''^2) + c^2(a'b' - c''^2)$$
$$+ 2bc(b''c'' - a'a'') + 2ca(c''a'' - b'b'') + 2ab(a''b'' - c'c'') = 0,$$

as the symmetrical form of the partial differential equation of developable surfaces.

2. A Tubular surface is the envelop of a series of spheres of invariable radius, the centres of which lie in a curve of which the equations are given. Let ρ be the radius of each sphere; α, β, γ, the co-ordinates of the centre of any one of the spheres: then, x, y, z, being the co-ordinates of any point of the envelop, we shall have

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = \rho^2 \ldots (1),$$

$$(x - \alpha)\,d\alpha + (y - \beta)\,d\beta + (z - \gamma)\,d\gamma = 0 \ldots (2).$$

The quantities α, β, γ, are subject to two equations

$$F(\alpha, \beta, \gamma) = 0, \quad f(\alpha, \beta, \gamma) = 0 \ldots\ldots (3).$$

From (1) and (2) we get

$$(x - \alpha)\,dx + (y - \beta)\,dy + (z - \gamma)\,dz = 0 \ldots (4).$$

Suppose $u = 0$ to be the equation to the tubular surface; then we shall have also

$$adx + bdy + cdz = 0 \ldots\ldots\ldots\ldots (5).$$

Now α, β, γ, x, y, z, being connected by the equations (1), (2), (3), it is evident that α, β, γ, z, may be regarded as functions of two independent variables x and y: we have then, from (4) and (5), by the aid of an indeterminate multiplier λ,

$$\lambda a + x - \alpha = 0, \quad \lambda b + y - \beta = 0, \quad \lambda c + z - \gamma = 0 \ldots (6).$$

Now the only equations connecting α, β, γ, x, y, z, with $d\alpha$, $d\beta$, $d\gamma$, are (2) and the differentials of (3); but all these three equations are satisfied identically by putting

$$d\alpha = 0, \quad d\beta = 0, \quad d\gamma = 0,$$

without subjecting to any limitation the absolute or relative values of x, y, z, α, β, γ: differentiating then equations (6) on this hypothesis, we get

$$-d\lambda = \frac{\lambda da + dx}{a} = \frac{\lambda db + dy}{b} = \frac{\lambda dc + dz}{c},$$

and therefore, performing the differentiations,

$$(1 + \lambda a')\,dx + \lambda c''dy + \lambda b''dz = -\,ad\lambda,$$
$$(1 + \lambda b')\,dy + \lambda a''dz + \lambda c''dx = -\,bd\lambda,$$
$$(1 + \lambda c')\,dz + \lambda b''dx + \lambda a''dy = -\,cd\lambda.$$

Eliminating dy and dz from these three equations, we get

$$-\frac{Vdx}{d\lambda} = a + \lambda\{a(b' + c') - bc'' - cb''\} + \lambda^2\{a(b'c' - a''^2)$$
$$+ a''(bb'' + cc'') - bc'c'' - cb'b''\} \ldots (7),$$

where $V = (1 + \lambda a')(1 + \lambda b')(1 + \lambda c') - \lambda^2(a''^2 + b''^2 + c''^2)$

$$-\lambda^3(a'a''^2 + b'b''^2 + c'c''^2 - 2a''b''c''),$$

a symmetrical function of a', b', c', a'', b'', c''. We must have, therefore, also

$$-\frac{Vdy}{d\lambda} = b + \lambda\{b(c' + a') - ca'' - ac''\} + \lambda^2\{b(c'a' - b''^2)$$
$$+ b''(cc'' + aa'') - ca'a'' - ac'c''\} \ldots (8),$$
$$-\frac{Vdz}{d\lambda} = c + \lambda\{c(a' + b') - ab'' - ba''\} + \lambda^2\{c(a'b' - c''^2)$$
$$+ c''(aa'' + bb'') - ab'b'' - ba'a''\} \ldots (9).$$

Multiplying equations (7), (8), (9), by a, b, c, respectively, adding, and paying attention to (5), we get

$$0 = a^2 + b^2 + c^2 + \lambda\{a'(b^2 + c^2) + b'(c^2 + a^2) + c'(a^2 + b^2)$$
$$- 2a''bc - 2b''ca - 2c''ab\}$$
$$+ \lambda^2\{a^2(b'c' - a''^2) + b^2(c'a' - b''^2) + c^2(a'b' - c''^2)$$
$$+ 2bc(b''c'' - a'a'') + 2ca(c''a'' - b'b'') + 2ab(a''b'' - c'c'')\} = 0:$$

but, from (1) and (6),

$$\lambda^2 = \frac{\rho^2}{a^2 + b^2 + c^2};$$

hence we obtain for the symmetrical form of the differential equation to tubular surfaces,

$$\begin{aligned}(a^2 + b^2 + c^2)^2 \pm \rho\,(a^2 + b^2 + c^2)^{\frac{1}{2}}\{a'\,(b^2 + c^2) + b'\,(c^2 + a^2) + c'\,(a^2 + b^2) \\ - 2a''bc - 2b''ca - 2c''ab\} \\ + \rho^2\,\{a^2(b'c' - a''^2) + b^2(c'a' - b''^2) + c^2(a'b' - c''^2) \\ + 2bc\,(b''c'' - a'a'') + 2ca\,(c''a'' - b'b'') + 2ab\,(a''b'' - c'c'')\} = 0.\end{aligned}$$

3. Let the equation to any surface be

$$f = f(x, y, z, a, \beta, \gamma) = 0 \;\ldots\ldots\ldots\ldots (1),$$

and let it be proposed to find the partial differential equation to its envelop, when the parameters a, β, γ, vary under the conditions expressed by two equations

$$\phi(a, \beta, \gamma) = 0, \quad \psi(a, \beta, \gamma) = 0 \;\ldots\ldots (2),$$

the forms of ϕ and ψ not being assigned.

We shall have, also,

$$\frac{df}{da}\,da + \frac{df}{d\beta}\,d\beta + \frac{df}{d\gamma}\,d\gamma = 0 \;\ldots\ldots\ldots (3),$$

and therefore, from (1),

$$\frac{df}{dx}\,dx + \frac{df}{dy}\,dy + \frac{df}{dz}\,dz = 0 \ldots\ldots\ldots\ldots (4).$$

Suppose $u = 0$ to be the equation to the envelop; then

$$adx + bdy + cdz = 0 \;\ldots\ldots\ldots\ldots (5).$$

In what follows we shall put

$$\frac{df}{dx} = l, \qquad \frac{df}{dy} = m, \qquad \frac{df}{dz} = n,$$

$$\frac{d^2f}{dx^2} = l', \qquad \frac{d^2f}{dy^2} = m', \qquad \frac{d^2f}{dz^2} = n',$$

$$\frac{d^2f}{dydz} = l'', \qquad \frac{d^2f}{dzdx} = m'', \qquad \frac{d^2f}{dxdy} = n''.$$

Now a, β, γ, x, y, z, being connected by the equations (1), (2), (3), it is evident that a, β, γ, z, may be regarded as functions of two independent variables x and y: we have then, from (4) and (5), by the aid of an indeterminate multiplier λ,

$$\lambda a + l = 0, \quad \lambda b + m = 0, \quad \lambda c + n = 0 \ldots\ldots (6).$$

Now the only equations connecting $\alpha, \beta, \gamma, x, y, z$, with $d\alpha$, $d\beta$, $d\gamma$, are (3) and the differentials of (2); but all these three equations are satisfied identically by putting

$$d\alpha = 0, \quad d\beta = 0, \quad d\gamma = 0,$$

without subjecting to any limitation the absolute or relative values of $x, y, z, \alpha, \beta, \gamma$: differentiating, then, equations (6) on this hypothesis, we get

$$(\lambda a' + l')\,dx + (\lambda c'' + n'')\,dy + (\lambda b'' + m'')\,dz = -\,a d\lambda,$$
$$(\lambda b' + m')\,dy + (\lambda a'' + l'')\,dz + (\lambda c'' + n'')\,dx = -\,b d\lambda,$$
$$(\lambda c' + n')\,dz + (\lambda b'' + m'')\,dx + (\lambda a'' + l'')\,dy = -\,c d\lambda.$$

Eliminating dy and dz from these equations by cross-multiplication, we get

$$\begin{aligned}-\frac{Vdx}{d\lambda} &= a\left\{(\lambda b' + m')(\lambda c' + n') - (\lambda a'' + l'')^2\right\}\\ &+ b\left\{(\lambda a'' + l'')(\lambda b'' + m'') - (\lambda c' + n')(\lambda c'' + n'')\right\}\\ &+ c\left\{(\lambda c'' + n'')(\lambda a'' + l'') - (\lambda b' + m')(\lambda b'' + m'')\right\} \ldots(7),\end{aligned}$$

where $V = (\lambda a' + l')(\lambda b' + m')(\lambda c' + n') + 2(\lambda a'' + l'')(\lambda b'' + m'')(\lambda c'' + n'')$
$\quad - (\lambda a' + l')(\lambda a'' + l'')^2 - (\lambda b' + m')(\lambda b'' + m'')^2 - (\lambda c' + n')(\lambda c'' + n'')^2$,

which is a symmetrical function of

$$\lambda a' + l', \quad \lambda b' + m', \quad \lambda c' + n',$$
$$\lambda a'' + l'', \quad \lambda b'' + m'', \quad \lambda c'' + n'':$$

we must, therefore, have also

$$\begin{aligned}-\frac{Vdy}{d\lambda} &= b\left\{(\lambda c' + n')(\lambda a' + l') - (\lambda b'' + m'')^2\right\}\\ &+ c\left\{(\lambda b'' + m'')(\lambda c'' + n'') - (\lambda a' + l')(\lambda a'' + l'')\right\}\\ &+ a\left\{(\lambda a'' + l'')(\lambda b'' + m'') - (\lambda c' + n')(\lambda c'' + n'')\right\} \ldots(8),\end{aligned}$$

$$\begin{aligned}-\frac{Vdz}{d\lambda} &= c\left\{(\lambda a' + l')(\lambda b' + m') - (\lambda c'' + n'')^2\right\}\\ &+ a\left\{(\lambda c'' + n'')(\lambda a'' + l'') - (\lambda b' + m')(\lambda b'' + m'')\right\}\\ &+ b\left\{(\lambda b'' + m'')(\lambda c'' + n'') - (\lambda a' + l')(\lambda a'' + l'')\right\} \ldots(9).\end{aligned}$$

Multiplying (7), (8), (9), by a, b, c, respectively, adding, and regarding (5), we obtain

$$\begin{aligned}0 &= a^2\left\{(\lambda b' + m')(\lambda c' + n') - (\lambda a'' + l'')^2\right\}\\ &+ b^2\left\{(\lambda c' + n')(\lambda a' + l') - (\lambda b'' + m'')^2\right\}\\ &+ c^2\left\{(\lambda a' + l')(\lambda b' + m') - (\lambda c'' + n'')^2\right\}\\ &+ 2bc\left\{(\lambda b'' + m'')(\lambda c'' + n'') - (\lambda a' + l')(\lambda a'' + l'')\right\}\\ &+ 2ca\left\{(\lambda c'' + n'')(\lambda a'' + l'') - (\lambda b' + m')(\lambda b'' + m'')\right\}\\ &+ 2ab\left\{(\lambda a'' + l'')(\lambda b'' + m'') - (\lambda c' + n')(\lambda c'' + n'')\right\} \ldots(10)\end{aligned}$$

Now, from equations (1) and (6), we may determine α, β, γ, λ, in terms of x, y, z, a, b, c: substituting the resulting expressions for these four quantities in the equation (10), we shall thus obtain the partial differential equation to the envelop of surface (1).

4. The transformation of the partial differential equations from the symmetrical to the unsymmetrical form is readily effected. Suppose, in fact, the equation $u = 0$ to be reduced to the form

$$u = z - f(x, y) = 0:$$

then it is easily seen that, p, q, r, s, t, denoting the partial differential coefficients of z with respect to x and y according to the usual notation,

$$\begin{array}{lll} a = -p, & b = -q, & c = 1, \\ a' = -r, & b' = -t, & c' = 0, \\ a'' = 0, & b'' = 0, & c'' = -s. \end{array}$$

If we substitute these values of the partial differential coefficients of u in the partial differential equations to the surface, we shall at once effect the proposed transformation. Thus the equation to developable surfaces becomes

$$rt - s^2 = 0;$$

and the equation to tubular surfaces assumes the form

$$(1+p^2+q^2)^2 \pm \rho(1+p^2+q^2)^{\frac{1}{2}}\{r(1+q^2) - 2pqs + t(1+p^2)\} + \rho^2(rt-s^2) = 0,$$

which may be seen in Moigno's *Leçons de Calcul Differential et de Calcul Integral*, tom. I. p. 478.

III.—ON THE AXIS OF SPONTANEOUS ROTATION.

When a rigid system is suddenly put in motion by the action of impulsive forces, there will under certain circumstances be a straight line, about which the system will begin to revolve as an instantaneous axis; this line is called the axis of Spontaneous Rotation. I have nowhere seen any investigation of the condition to be satisfied, in order that such an axis may exist; and this is what I now propose to supply.

Let the rigid system be subject to impulsive forces, which are reducible to three impulsive pressures, X, Y, Z, at the origin, and three impulsive couples whose moments are L, M, N.

Let V_x, V_y, V_z be the absolute velocities of a particle δm, (whose co-ordinates, measured from the centre of gravity, are x, y, z,) parallel to the axes of co-ordinates; V_x', V_y', V_z' the

velocities of the same point relative to the centre of gravity; $\overline{V}_x, \overline{V}_y, \overline{V}_z$ the velocities of the centre of gravity; $\omega_1, \omega_2, \omega_3$ the impulsive angular velocities of the system about the three axes.

Then we have these relations,

$$V_x = \overline{V}_x + V'_x, \quad V_y = \overline{V}_y + V'_y, \quad V_z = \overline{V}_z + V'_z;$$

and

$$\left.\begin{aligned} V'_x &= z\omega_2 - y\omega_3 \\ V'_y &= x\omega_3 - z\omega_1 \\ V'_z &= y\omega_1 - x\omega_2 \end{aligned}\right\} \ldots\ldots\ldots\ldots (1).$$

Now we have for the motion about the centre of gravity the equations

$$\left.\begin{aligned} \Sigma\delta m(yV'_z - zV'_y) &= L \\ \Sigma\delta m(zV'_x - xV'_z) &= M \\ \Sigma\delta m(xV'_y - yV'_x) &= N \end{aligned}\right\} \ldots\ldots\ldots\ldots (2);$$

which, by substituting for V'_x, V'_y, V'_z the values given above, become

$$\left.\begin{aligned} \omega_1\, \Sigma\delta m(y^2 + z^2) - \omega_2\, \Sigma\delta mxy - \omega_3\, \Sigma\delta mxz &= L \\ \omega_2\, \Sigma\delta m(x^2 + z^2) - \omega_3\, \Sigma\delta myz - \omega_1\, \Sigma\delta mxy &= M \\ \omega_3\, \Sigma\delta m(x^2 + y^2) - \omega_1\, \Sigma\delta mxz - \omega_2\, \Sigma\delta myz &= N \end{aligned}\right\} \ldots (3).$$

To simplify these equations, suppose the principal axes through the centre of gravity to be axes of co-ordinates, which may always be done without loss of generality, and call A, B, C the principal moments of inertia of the system; then we have

$$\left.\begin{aligned} \omega_1 &= \frac{L}{A} \\ \omega_2 &= \frac{M}{B} \\ \omega_3 &= \frac{N}{C} \end{aligned}\right\} \ldots\ldots\ldots\ldots (4).$$

Again, we have for the motion of the centre of gravity, (if m be the whole mass of the system,)

$$\left.\begin{aligned} \overline{V}_x &= \frac{X}{m} \\ \overline{V}_y &= \frac{Y}{m} \\ \overline{V}_z &= \frac{Z}{m} \end{aligned}\right\} \ldots\ldots\ldots\ldots (5);$$

and therefore, for the absolute velocities of any particle δm, we shall have

$$\left.\begin{aligned} V_x &= \overline{V}_x + V_x' = \frac{X}{m} + z\frac{M}{B} - y\frac{N}{C} \\ V_y &= \overline{V}_y + V_y' = \frac{Y}{m} + x\frac{N}{C} - z\frac{L}{A} \\ V_z &= \overline{V}_z + V_z' = \frac{Z}{m} + y\frac{L}{A} - x\frac{M}{B} \end{aligned}\right\} \text{..........} (6).$$

To determine the points which are at rest, we must put $V_x = 0$, $V_y = 0$, $V_z = 0$, and we have

$$\left.\begin{aligned} y\frac{N}{C} - z\frac{M}{B} &= \frac{X}{m} \\ z\frac{L}{A} - x\frac{N}{C} &= \frac{Y}{m} \\ x\frac{M}{B} - y\frac{L}{A} &= \frac{Z}{m} \end{aligned}\right\} \text{.............} (7);$$

these three equations are not independent, for if we multiply them by $\frac{L}{A}$, $\frac{M}{B}$ and $\frac{N}{C}$ respectively and add, there results

$$\frac{L.X}{A} + \frac{M.Y}{B} + \frac{N.Z}{C} = 0 \text{ .... } (8),$$

which is a relation between the forces and the constitution of the system, in order that there may be an axis of spontaneous rotation; and if this condition be satisfied, any two of the equations (7) will be the equations to the axis.

If the moments of inertia A, B, C, be all equal, the equation (8) becomes the condition of the forces acting on the system, having a single resultant.

It is easy to shew that the axis of spontaneous rotation is perpendicular to the direction of the resultant of the forces; for the direction-cosines of this resultant are

$$\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R};$$

and, if θ be the angle between it and the spontaneous axis, we have

$$\cos\theta = \frac{\frac{LX}{A} + \frac{MY}{B} + \frac{NZ}{C}}{R\sqrt{\left(\frac{L^2}{A^2} + \frac{M^2}{B^2} + \frac{N^2}{C^2}\right)}} = 0, \quad \text{by (8)};$$

and therefore θ is a right angle.

This proposition may be more elegantly proved thus: multiplying equations (7) by x, y, z, respectively, and adding, we have

$$xX + yY + zZ = 0,$$

which is the equation to a plane in which the spontaneous axis lies, and is perpendicular to the line whose direction-cosines are proportional to X, Y, and Z.

It is not difficult to see the physical interpretation of the condition (8). If we consider the motion of the system as made up of the motion of translation of the centre of gravity, and the rotation about the centre of gravity, then X, Y, Z, will be proportional to the direction-cosines of the path of the centre of gravity, and $\frac{L}{A}, \frac{M}{B}, \frac{N}{C}$, being respectively equal to ω_1, ω_2, ω_3, will be proportional to the direction-cosines of the instantaneous axis, about which the system revolves when it receives the impulse: consequently the equation (8) expresses that the motion of the centre of gravity is in the plane perpendicular to the axis of rotation, and it is quite clear, that unless this be case, it is impossible that the motion of any point due to the motion of the centre of gravity should be counteracted by the motion due to rotation about the centre of gravity. The equation (8) is, in fact, exactly similar to that which expresses the condition, that a system of forces shall admit of a single resultant: for the equation expressing this latter condition, implies that the resultant force acts in the plane of the resultant couple; or, which is the same thing, that the plane in which there is a tendency to translation is perpendicular to the axis about which there is a tendency to rotation.

In the particular case of a blow at a definite point, whose co-ordinates are a, b, c, we have

$$L = bZ - cY : M = cX - aZ : N = aY - bX,$$

and the condition (8) becomes

$$\frac{a}{X}\left(\frac{1}{B} - \frac{1}{C}\right) + \frac{b}{Y}\left(\frac{1}{C} - \frac{1}{A}\right) + \frac{c}{Z}\left(\frac{1}{A} - \frac{1}{B}\right) = 0 \ldots\ldots (9).$$

Let us suppose this blow to be parallel to one of the principal axes, as the axis of z for instance, and in the plane of xz; then $b = 0$, $c = 0$, $X = 0$, $Y = 0$, $L = 0$, $M = -aZ$, $N = 0$; and equations (7) become

$$z = 0,$$

$$x = -\frac{B}{ma} = -\frac{k^2}{a} \ldots\ldots\ldots (10),$$

if k be the radius of gyration with respect to the axis of x.

In this case, if the line of action of the blow were made the *axis of suspension,* the point in which the spontaneous axis meets the plane of xz, would be the *centre of oscillation* of the system. The axis thus determined is that found as the axis of Spontaneous Rotation in *Pratt's Mechanical Philosophy:* it may be seen from the preceding investigation, how very limited the application of the formula (10) must be.

It will be worth while to consider fully the case of a rigid system acted upon by a single blow: for this purpose, I shall assume the blow to act parallel to the axis of z, and in the plane of xy; but, to obtain the necessary generality, I shall no longer assume the principal axes to be axes of co-ordinates. Resuming then equations (3), and putting $\Sigma\, \delta m\, yz = D$, $\Sigma\, \delta m\, xz = E$, and $\Sigma\, \delta m\, xy = F$, we have, for the determination $\omega_1, \omega_2, \omega_3$,

$$\left.\begin{aligned} A\omega_1 - F\omega_2 - E\omega_3 &= 0 \\ B\omega_2 - D\omega_3 - F\omega_1 &= -aZ \\ C\omega_3 - E\omega_1 - D\omega_2 &= 0 \end{aligned}\right\} \ldots\ldots\ldots (11);$$

and for the equations to the spontaneous axis,

$$\left.\begin{aligned} y\omega_3 - z\omega_2 &= 0 \\ z\omega_1 - x\omega_3 &= 0 \\ x\omega_2 - y\omega_1 &= \frac{Z}{m} \end{aligned}\right\} \ldots\ldots\ldots\ldots (12).$$

Multiplying these last equations by ω_1, ω_2, ω_3 respectively, and adding, we see that ω_3 must $= 0$, which reduces equations (11) to the following,

$$\left.\begin{aligned} A\omega_1 - F\omega_2 &= 0 \\ B\omega_2 - F\omega_1 &= -aZ \\ E\omega_1 + D\omega_2 &= 0 \end{aligned}\right\} \ldots\ldots\ldots (13);$$

whence

$$\left.\begin{aligned} \omega_1 &= \frac{F}{F^2 - AB} \cdot aZ \\ \omega_2 &= \frac{A}{F^2 - AB} \cdot aZ \end{aligned}\right\} \ldots\ldots\ldots (14);$$

with the condition $\quad AD + EF = 0 \ldots\ldots\ldots\ldots (15)$,

and equations (12) become

$$\left.\begin{aligned} z &= 0, \\ Ax - Fy &= \frac{F^2 - AB}{ma} \end{aligned}\right\} \ldots\ldots\ldots (16).$$

On the whole, therefore, we have the condition (15), in order

that there may be a spontaneous axis; and if there be, its equations are (16).

The axis of spontaneous rotation possesses a remarkable property, the discovery of which is I believe due to Euler, and of which Lagrange has given a demonstration in the *Mécanique Analytique*; viz. that the *vis-viva* of the system with respect to this axis is a *maximum* or a *minimum*. Lagrange's proof leaves it doubtful which it is, but I believe it will appear that it is always the former. I shall proceed to obtain a general expression for the *vis-viva* of the system, from which also the truth of the above proposition may be made to appear.

By the general equation of *vis-viva*, we have

$$\begin{aligned} \textit{vis-viva} &= \Sigma\,(XV_x + YV_y + ZV_z) \\ &= \Sigma\,\{X(\overline{V}_x + z\omega_2 - y\omega_3) + Y(\overline{V}_y + x\omega_3 - z\omega_1) \\ &\qquad\qquad + Z(\overline{V}_z + y\omega_1 - x\omega_2)\} \\ &= \Sigma\,(X\overline{V}_x + Y\overline{V}_y + Z\overline{V}_z) + \Sigma\,(L\omega_1 + M\omega_2 + N\omega_3) \\ &= \frac{X^2 + Y^2 + Z^2}{m} + \frac{L^2}{A} + \frac{M^2}{B} + \frac{N^2}{C} \;\ldots\ldots\; (17). \end{aligned}$$

(The Σ is dropped, because X, Y, Z, L, M, N, are supposed to be the resultants of all the forces and couples acting on the system.)

From this equation, which involves no elements of space but such as depend on the system itself, we conclude that the whole *vis-viva* generated by the impulse is an absolute constant. If now we were to consider any axis in the system fixed, there would be an impulsive pressure on this axis which would *destroy vis-viva;* but, by the very nature of the axis of Spontaneous Rotation, there is no impulsive pressure upon it; consequently, if it be fixed, no *vis-viva* is destroyed, and therefore the *vis-viva* calculated with regard to this axis will be greater than for any other. H. G.

IV.—ON BRIANCHON'S HEXAGON.

By W. Walton, M.A. Trin. Coll.

If any hexagon be circumscribed about a conic section, the three diagonals joining opposite angles will all pass through one point.

This elegant property of conic sections was first given by M. Brianchon, in the 13th *Cahier* of the *Journal de l'Ecole Polytechnique*, p. 301, where it is deduced as a corollary

from Pascal's *Hexagramme Mystique.* An analytical demonstration of this theorem in the particular case of the parabola, has been supplied by Mr. Lubbock, in the *Philosophical Magazine* for August 1838: another demonstration for the same case may be seen in the *Cambridge Mathematical Journal* for February 1839. I am not aware that up to the present time any purely algebraical demonstrations have been given for the cases of the ellipse and hyperbola. To supply this deficiency is the object of this paper. I shall first give a demonstration for the case of the hyperbola, and afterwards one for the ellipse, which, as will be seen, likewise includes another demonstration for the hyperbola.

I. The equation to the hyperbola referred to its asymptotes is

$$4xy = c^2.$$

Let the equation to a tangent-line be

$$\frac{x}{a_1} + \frac{y}{\beta_1} = 1:$$

then the roots of the equation

$$\frac{x^2}{a_1} - x + \frac{c^2}{4\beta_1} = 0,$$

which belong to the common point of the tangent and curve, must be equal: hence $a_1\beta_1 = c^2$.

Thus the equation to the tangent is

$$\frac{x}{a_1} + \frac{a_1 y}{c^2} = 1:$$

at the intersection of this tangent and another

$$\frac{x}{a_2} + \frac{a_2 y}{c^2} = 1,$$

we shall have $x_{1,2} = \dfrac{a_1 a_2}{a_1 + a_2}$, $y_{1,2} = \dfrac{c^2}{a_1 + a_2}$.

These will be the co-ordinates of one of the angles of the circumscribed hexagon, the two tangents being two of the sides. We shall have analogous expressions for the co-ordinates of the other angles.

The equation to the diagonal through the angles $(x_{1,2}, y_{1,2})$, $(x_{4,5}, y_{4,5})$ will be

$$x(y_{4,5} - y_{1,2}) - y(x_{4,5} - x_{1,2}) = x_{1,2}y_{4,5} - x_{4,5}y_{1,2},$$

or, if we substitute for the co-ordinates of the angular points their values,

$$c^2x\,\{(\alpha_4-\alpha_1)-(\alpha_2-\alpha_5)\}+y\,\{\alpha_4\alpha_1(\alpha_5-\alpha_2)+\alpha_5\alpha_2(\alpha_4-\alpha_1)\}$$
$$=c^2(\alpha_4\alpha_5-\alpha_1\alpha_2)\ldots(1).$$

Similarly the equation to the diagonal, through the angular points $(x_{2,3}, y_{2,3})$, $(x_{5,6}, y_{5,6})$, will be

$$c^2x\,\{(\alpha_5-\alpha_2)-(\alpha_3-\alpha_6)\}+y\,\{\alpha_5\alpha_2(\alpha_6-\alpha_3)+\alpha_6\alpha_3(\alpha_5-\alpha_2)\}$$
$$=c^2(\alpha_5\alpha_6-\alpha_2\alpha_3)\ldots(2).$$

At the intersection of these two diagonals, multiplying the equation (1) by $\alpha_3-\alpha_6$, and the equation (2) by $\alpha_1-\alpha_4$, subtracting the latter of the resulting equations from the former, and dividing the final equation by $\alpha_2-\alpha_5$, we shall get

$$c^2x\,\{(\alpha_6-\alpha_3)-(\alpha_4-\alpha_1)\}+y\,\{\alpha_6\alpha_3(\alpha_1-\alpha_4)+\alpha_1\alpha_4(\alpha_6-\alpha_3)\}$$
$$=c^2(\alpha_6\alpha_1-\alpha_3\alpha_4)\ldots(3).$$

But, as is evident from symmetry, equation (3) belongs to the third diagonal, namely, that which passes through the points $(x_{3,4}, y_{3,4})$, $(x_{6,1}, y_{6,1})$. Thus we see that the two diagonals (1) and (2) intersect in the third; which establishes the theorem.

Cor. Subtracting the sum of (1) and (3) from (2), we get for the value of y, at the point through which the three diagonals pass,

$$y\,\{\alpha_1\alpha_2(\alpha_4+\alpha_5)-\alpha_2\alpha_3(\alpha_5+\alpha_6)+\alpha_3\alpha_4(\alpha_6+\alpha_1)-\alpha_4\alpha_5(\alpha_1+\alpha_2)$$
$$+\alpha_5\alpha_6(\alpha_2+\alpha_3)-\alpha_6\alpha_1(\alpha_3+\alpha_4)\}=c^2(\alpha_1\alpha_2-\alpha_2\alpha_3+\alpha_3\alpha_4-\alpha_4\alpha_5+\alpha_5\alpha_6-\alpha_6\alpha_1).$$

The value of x, by virtue of symmetry, will be

$$x\,\{\beta_1\beta_2(\beta_4+\beta_5)-\beta_2\beta_3(\beta_5+\beta_6)+\beta_3\beta_4(\beta_6+\beta_1)-\beta_4\beta_5(\beta_1+\beta_2)$$
$$+\beta_5\beta_6(\beta_2+\beta_3)-\beta_6\beta_1(\beta_3+\beta_4)\}$$
$$=c^2(\beta_1\beta_2-\beta_2\beta_3+\beta_3\beta_4-\beta_4\beta_5+\beta_5\beta_6-\beta_6\beta_1).$$

II. The equation to the tangent of an ellipse is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1\ \ldots\ldots\ldots\ldots\ (1),$$

x' and y' being connected by the equation

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1\ldots\ldots\ldots\ldots\ldots(2).$$

Equation (2) is equivalent to the following system of equations,

$$\frac{x'}{a}+\frac{y'}{b}\sqrt{(-1)}=\alpha_1,$$

$$\frac{x'}{a}-\frac{y'}{b}\sqrt{(-1)}=\frac{1}{\alpha_1},$$

a_1 being an arbitrary quantity: hence

$$x' = a\left(a_1 + \frac{1}{a_1}\right), \quad y' = \frac{b}{\sqrt{(-1)}}\left(a_1 - \frac{1}{a_1}\right):$$

hence, equation (1) becomes

$$\frac{x}{a}\left(a_1 + \frac{1}{a_1}\right) + \frac{y}{b\sqrt{(-1)}}\left(a_1 - \frac{1}{a_1}\right) = 1.$$

Put $a = 2a'$, $b\sqrt{(-1)} = 2b'$; and the equation to the tangent, which we may suppose to be one of the sides of the hexagon, assumes the form

$$\frac{x}{2a'}\left(a_1 + \frac{1}{a_1}\right) + \frac{y}{2b'}\left(a_1 - \frac{1}{a_1}\right) = 1 \ldots\ldots\ldots (3).$$

The equation to the next side of the hexagon will be, in like manner, of the form

$$\frac{x}{2a'}\left(a_2 + \frac{1}{a_2}\right) + \frac{y}{2b'}\left(a_2 - \frac{1}{a_2}\right) = 1 \ \ldots\ldots (4).$$

At the intersection of (3) and (4), $x_{1,2}$, $y_{1,2}$ being the co-ordinates of the corresponding angle of the hexagon, we shall get, by the combination of the equations,

$$x_{1,2} = a'\frac{1 + a_1a_2}{a_1 + a_2}, \quad y_{1,2} = b'\frac{1 - a_1a_2}{a_1 + a_2}.$$

The equation to the diagonal, through the angular points $(x_{1,2}, y_{1,2})$, $(x_{4,5}, y_{4,5})$, will be

$$x(y_{4,5} - y_{1,2}) - y(x_{4,5} - x_{1,2}) = x_{1,2}y_{4,5} - x_{4,5}y_{1,2};$$

or, substituting for the co-ordinates of the angular points their values,

$$\begin{aligned} b'x\,\{(a_1 - a_4)(1 + a_2a_5) + (a_2 - a_5)(1 + a_1a_4)\} - a'y\,\{(a_1 - a_4)(1 - a_2a_5) \\ + (a_2 - a_5)(1 - a_1a_4)\} = 2a'b'(a_1a_2 - a_4a_5) \ldots (5). \end{aligned}$$

By similarity, it is evident that the equation to the diagonal, through $(x_{2,3}, y_{2,3})$, $(x_{5,6}, y_{5,6})$, will be

$$\begin{aligned} b'x\,\{(a_2 - a_5)(1 + a_3a_6) + (a_3 - a_6)(1 + a_2a_5)\} - a'y\,\{(a_2 - a_5)(1 - a_3a_6) \\ + (a_3 - a_6)(1 - a_2a_5)\} = 2a'b'(a_2a_3 - a_5a_6) \ldots (6). \end{aligned}$$

At the intersection of these two diagonals, multiplying (5) by $a_3 - a_6$, (6) by $a_1 - a_4$, subtracting the latter of the resulting equations from the former, and dividing the final equation by $a_2 - a_5$, we shall thus get

$$\begin{aligned} b'x\,\{(a_3 - a_6)(1 + a_4a_1) + (a_4 - a_1)(1 + a_3a_6)\} - a'y\,\{(a_3 - a_6)(1 - a_4a_1) \\ + (a_4 - a_1)(1 - a_3a_6)\} = 2a'b'(a_3a_4 - a_6a_1) \ldots (7). \end{aligned}$$

But by the symmetry we know that (7) is the equation to the third diagonal; hence the diagonals (5), (6), intersect in (7).

III. The demonstration which we have given for the ellipse really comprehends a proof for the hyperbola. In fact the equation to the tangent at any point of an hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

may be expressed in the form

$$\frac{x}{2a'}\left(a_1 + \frac{1}{a_1}\right) + \frac{y}{2b'}\left(a_1 - \frac{1}{a_1}\right) = 1,$$

by the process given for the ellipse. In the case of the hyperbola $2a' = a$, $2b' = -b$. Thus the demonstration for the ellipse coincides with that for the hyperbola.

V.—NOTES ON LINEAR TRANSFORMATIONS.

By George Boole.

1. The complete solution of the problem of which the object is to take away the products of the variables from a homogeneous function of the second degree,

$$ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy \ldots\ldots (1),$$

requires the determination of the coefficients α, β, γ, &c. in the linear theorems

$$x = \alpha x' + \beta y' + \gamma z' \ldots\ldots\ldots\ldots (2),$$

$$y = \alpha' x' + \beta' y' + \gamma' z' \ldots\ldots\ldots\ldots (3),$$

$$z = \alpha'' x' + \beta'' y' + \gamma'' z' \ldots\ldots\ldots\ldots (4),$$

as well as of a', b', c', in the transformed function

$$a'x'^2 + b'y'^2 + c'z'^2 \ldots\ldots\ldots\ldots (5).$$

The following investigation is intended to effect this object.

The transformation being supposed to be from one rectangular system of co-ordinates to another, we shall have, on squaring (2), the following system of equations of the second degree,

$$\left.\begin{array}{r} x^2 = \alpha^2 x'^2 + \beta^2 y'^2 + \gamma^2 z'^2 + 2\beta\gamma y'z' + 2\gamma\alpha z'x' + 2\alpha\beta x'y' \\ x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 \\ ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy = a'x'^2 + b'y'^2 + c'z'^2 \end{array}\right\} \ldots (6).$$

Treating this system of equations by the method given in pp. 108–114 of the *Mathematical Journal,* we obtain six

equations among the constants. Three of these equations are included in the cubic

$$(\eta - a)(\eta - b)(\eta - c) - d(\eta - a) - e(\eta - b) - f(\eta - c) + 2def = 0,$$

in which the values of η determine a', b', c'. The other three equations are

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \ldots\ldots\ldots\ldots\ldots (7),$$

$$(b' + c')\alpha^2 + (c' + a')\beta^2 + (a' + b')\gamma^2 = b + c \ldots (8),$$

$$b'c'\alpha^2 + c'a'\beta^2 + a'b'\gamma^2 = bc - d^2 \ldots\ldots\ldots (9).$$

Hence (7) $\times a'^2 -$ (8) $\times a' +$ (9) gives

$$(a'^2 + b'c' - a'b' - a'c')\alpha^2 = a'^2 - (b + c)a' + bc - d^2,$$

$$\therefore \alpha^2 = \frac{(a' - b)(a' - c) - d^2}{(a' - b')(a' - c')};$$

whence, by inspection,

$$\beta^2 = \frac{(b' - b)(b' - c) - d^2}{(b' - c')(b' - a')}, \quad \gamma^2 = \frac{(c' - b)(c' - c) - d^2}{(c' - a')(c' - b')}.$$

To obtain $\alpha'^2, \beta'^2, \gamma'^2$, we must change in the above, b, c, d into c, a, e, respectively; and to find $\alpha''^2, \beta''^2, \gamma''^2$, we must change b, c, d into a, b, f, respectively.

2. *Attraction of an Ellipsoid.* In investigating the attraction of the ellipsoid whose surface is defined by the equation

$$\frac{x^2}{h^2} + \frac{y^2}{h_1^2} + \frac{z^2}{h_2^2} = 1$$

on an external point, a, b, c, the force varying as $\frac{1}{d^2}$, we meet with an equation of the form

$$\frac{a^2}{\eta^2 + h^2} + \frac{b^2}{\eta^2 + h_1^2} + \frac{c^2}{\eta^2 + h_2^2} = 1.$$

It may be worth while to observe that the values of η^2, as determined by the above, are respectively equal to those which we should find for A, B, C, in transforming the homogeneous function

$$(a^2 - h^2)x^2 + (b^2 - h_1^2)y^2 + (c^2 - h_2^2)z^2 + 2bcyz + 2cezx + 2abxy$$

into the form

$$Ax'^2 + By'^2 + Cz'^2,$$

x, y, z, and x', y', z', denoting rectangular systems of co-ordinates.

3. Theorem. *If Q be a homogeneous function of the nth degree with m variables, $x_1, x_2 . . x_m$, which is transformed into*

R, a similar homogeneous function, by the relations

$$\left.\begin{aligned} x_1 &= \lambda_1 y_1 + \lambda_2 y_2 \ldots + \lambda_m y_m \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_m &= \rho_1 y_1 + \rho_2 y_2 \ldots + \rho_m y_m \end{aligned}\right\} \ldots\ldots\ldots\ldots (1);$$

and if γ be the degree of $\theta(Q)$ and $\theta(R)$, then

$$\theta(Q) = \frac{\theta(R)}{E^{\frac{\gamma n}{m}}} \ldots\ldots\ldots\ldots\ldots (2),$$

where E is the result obtained by eliminating the variables from the second members of (1) equated to 0.

The above theorem was given, but without demonstration, in vol. III. p. 19 of the *Mathematical Journal.* The following is the analysis by which it was obtained.

Let $q = r$ represent another equation analogous to the equation $Q = R$, q being a homogeneous function of $x_1 \ldots x_m$ of the n^{th} degree, r a similar function of $y_1, y_2 \ldots y_m$. By the theory of linear transformations (vol. III. p. 9), the equations $\theta(Q + hq) = 0$, $\theta(R + hr) = 0$, are identical relatively to h. If we suppose $\theta(Q + hq)$ expanded in ascending powers of h, we shall have an equation of the form

$$L + Mh \ldots + Zh^{\gamma} = 0,$$

wherein $L = \theta(Q)$, and Z, the last coefficient, $= \theta(q)$. In like manner $\theta(R + hr) = 0$ will assume the form

$$L_1 + M_1 h \ldots + Z_1 h^{\gamma} = 0,$$

in which $L_1 = \theta(R)$, $Z_1 = \theta(r)$. That the two equations may give identical values of h, a series of conditions must be fulfilled, of which the last is

$$\frac{L}{Z} = \frac{L_1}{Z_1},$$

wherefore

$$\frac{\theta(Q)}{\theta(q)} = \frac{\theta(R)}{\theta(r)} \ldots\ldots\ldots\ldots\ldots (3).$$

Now let $x_1, x_2 \ldots x_m$ vanish, then (1) gives

$$\left.\begin{aligned} \lambda_1 y_1 \ldots + \lambda_m y_m &= 0 \\ \ldots\ldots\ldots\ldots& \\ \rho_1 y_1 \ldots + \rho_m y_m &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (4).$$

Eliminating the variables, we have $E = 0$, which is the necessary and sufficient condition, that $x_1 \ldots x_m$ may vanish without causing $y_1 \ldots y_m$ to vanish.

Now, vol. III. p. 4, the condition

$$\frac{dq}{dx_1} = 0, \quad \frac{dq}{dx_2} = 0 \ldots \frac{dq}{dx_m} = 0 \ldots\ldots\ldots (5)$$

induces as a necessary consequence the condition

$$\frac{dr}{dy_1} = 0, \quad \frac{dr}{dy_2} = 0 \ldots \frac{dr}{dy_m} = 0 \ldots\ldots\ldots (6).$$

Assume $q = x_1^n + x_2^n \ldots + x^n_m$, then

$$\frac{dq}{dx_1} = nx_1^{n-1} \ldots \frac{dq}{dx_m} = nx_m^{n-1}.$$

Hence, if $x_1, x_2 \ldots x_m$, vanish, the condition (5) is satisfied, and therefore (6). Eliminating the variables from (6), we have $\theta(r) = 0$, the necessary and sufficient condition in order that $x_1, x_2 \ldots x_m$, may vanish without causing $y_1, y_2 \ldots y_m$, to vanish also.

Since, from the above, $\theta(r)$ and E are so related that the one cannot vanish without causing the other to vanish also, they must be connected by an equation of the form

$$\theta(r) = CE^\lambda \ldots\ldots\ldots\ldots\ldots (7),$$

C being a constant, and λ a positive constant to be determined. Both C and λ, it is to be observed, are quite independent of $\lambda_1 \ldots \lambda_m$, $\rho_1 \ldots \rho_m$, the constants in the linear theorems.

If in q we substitute for $x_1, x_2 \ldots x_m$, their values from (1) in terms of $y_1, y_2 \ldots y_m$, the resulting coefficients of r will each be of the n^{th} degree in terms of λ_1, ρ_1, &c.; wherefore $\theta(r)$ is of the $(\gamma n)^{\text{th}}$ degree with respect to $\lambda_1 \ldots \rho_1$, &c. Now E is of the m^{th} degree with respect to those quantities, wherefore $\theta(r)$ is of the $\left(\gamma \frac{n}{m}\right)^{\text{th}}$ degree with respect to E, and λ in $(7) = \gamma \frac{n}{m}$.

As C is quite independent of $\lambda_1 \ldots \rho_1$, &c., we may, in determining C, attribute to those quantities what values we please. Assume them such that we may have

$$x_1 = y_1, \quad x_2 = y_2 \ldots x_m = y_m,$$

and let r_1 be the particular value of r under these circumstances, then $\theta(r_1) = \theta(q)$, $E = 1$, wherefore, by (7),

$$C = \theta(r_1) = \theta(q),$$

whence, substituting in (7) for C and λ,

$$\theta(r) = \theta(q) E^{\frac{\gamma n}{m}}.$$

Employing this value of $\theta(r)$ in (3), we have

$$\theta(Q) = \frac{\theta(R)}{E^{\frac{\gamma n}{m}}}.$$

From a formula given in one of Professor Sylvester's papers on Elimination, in the *Philosophical Magazine*, it appears, that $\gamma = m(n-1)^{m-1}$, so that the above would give

$$\theta(Q) = \frac{\theta(R)}{E^{n(n-1)^{m-1}}}.$$

4. *Determination of $\theta(Q)$ when Q is a homogeneous function of the fourth degree with two variables.*

When Q is of the form $ax^4 + 4bx^3y + 6ex^2y^2 + 4dxy^3 + ey^4$, the value of $\theta(Q)$ is found by eliminating the variables from the equations

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0,$$
$$bx^3 + 3cx^2y + 3dxy^2 + ey^3 = 0.$$

After a tedious process, we find

$$\begin{aligned}\theta(Q) = a^3e^3 &- 6ab^2d^2e - 12a^2bde^2 - 18a^2c^2e^2 - 27a^2d^4 - 27b^4e^2\\ &+ 36b^2c^2d^2 + 54a^2cd^2e + 54ab^2ce^2 - 54ac^3d^2 - 54b^2c^3e - 64b^3d^3\\ &+ 81ac^4e + 108abcd^3 + 108b^3cde - 180abc^2de.\end{aligned}$$

The above result may probably be found to have some applications in the theory of the higher algebraic equations.

Lincoln, June, 1844.

VI.—ON A PROBLEM IN CENTRAL FORCES.

A PARTICLE moves about a centre of attractive force varying directly as the distance; to determine the motion, having given the velocity and direction of projection, and also the initial position of the particle.

The solution of this problem is ordinarily effected, either by means of the polar differential equation, or by resolving the force in directions parallel to two rectangular axes. The motion however may be more conveniently referred to a pair of oblique axes, selected as we shall explain in this paper.

Let the centre of force be taken as the origin of co-ordinates, and let the axis of x be chosen so as to pass through the initial position of the particle. Let the axis of y be taken parallel to the direction of projection. The co-ordinate axes will thus generally be oblique to each other.

For the motion there is, μ^2 denoting the absolute force,

$$\frac{d^2x}{dt^2} + \mu^2 x = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{d^2y}{dt^2} + \mu^2 y = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

The integral of equation (1) is

$$x = A \cos(\mu t + \epsilon),$$

A, ϵ, being arbitrary constants. Let a be the initial value of x; then

$$a = A \cos \epsilon;$$

and, since $\frac{dx}{dt}$ is initially equal to zero,

$$0 = -A\mu \sin \epsilon;$$

hence the integral becomes

$$x = a \cos(\mu t) \ldots\ldots\ldots\ldots\ldots (3).$$

The integral of (2) is

$$y = A' \cos(\mu t + \epsilon'),$$

A', ϵ', being arbitrary constants. Let v be the initial value of $\frac{dy}{dt}$; then, zero being the initial value of y,

$$0 = A' \cos \epsilon',$$

$$v = -A' \mu \sin \epsilon',$$

and therefore the integral becomes

$$y = \frac{v}{\mu} \sin(\mu t) \ldots\ldots\ldots\ldots (4).$$

From (3) and (4) we see that, putting $\frac{v}{\mu} = b$,

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right) = 1 \ldots\ldots\ldots\ldots (5),$$

which is the equation to an ellipse of which the centre coincides with the centre of force, the directions of x and y coinciding with the semi-conjugate diameters a, b.

From (3) and (4) we see that the values of x, y, $\frac{dx}{dt}$, $\frac{dy}{dt}$, will not be altered if t be so increased that μt increase by 2π; hence it follows that the periodic time is equal to $\frac{2\pi}{\mu}$.

Let r be the distance of any point in the path from the origin; then, ω being the angle between the axes,

$$r^2 = x^2 + y^2 + 2xy \cos \omega \ldots\ldots\ldots\ldots (6).$$

At the extremities of the axes of the ellipse, $dr = 0$; hence, for the determination of these points, we have, from (5) and (6),

$$0 = \frac{xdx}{a^2} + \frac{ydy}{b^2},$$

$$0 = (x + y \cos \omega)\, dx + (y + x \cos \omega)\, dy;$$

whence, λ being an arbitrary quantity,

$$\left(\frac{\lambda}{a^2} - 1\right) x = y \cos \omega \ldots\ldots\ldots\ldots (7),$$

$$\left(\frac{\lambda}{b^2} - 1\right) y = x \cos \omega \ldots\ldots\ldots\ldots (8).$$

From (7) and (8) there is, multiplying them by x, y, respectively, adding, and attending to (5) and (6),

$$\lambda = r^2 \ldots\ldots\ldots\ldots\ldots\ldots (9):$$

also, multiplying together (7) and (8), and dividing by xy, we have, by virtue of (9),

$$\left(\frac{r^2}{a^2} - 1\right)\left(\frac{r^2}{b^2} - 1\right) = \cos^2 \omega \ldots\ldots\ldots (10),$$

which gives two values of r^2, the one belonging to the semi-axis major, and the other to the semi-axis minor.

The equations to the semi-axes, r', r'', denoting the two roots of (10), are

$$\left(\frac{r'^2}{a^2} - 1\right) x = y \cos \omega,$$

$$\left(\frac{r''^2}{a^2} - 1\right) x = y \cos \omega:$$

or, which, by virtue of (10), comes to the same thing,

$$\left(\frac{r'^2}{b^2} - 1\right) y = x \cos \omega,$$

$$\left(\frac{r''^2}{b^2} - 1\right) y = x \cos \omega.$$

W. W.

VII.—NOTE ON THE SPONTANEOUS AXIS OF ROTATION.

In Article III. of this number of the *Journal*, on the Spontaneous Axis of Rotation, it is shewn that, under certain circumstances, expressed analytically by the equation (8), if

a free rigid body be struck by impulsive forces, there will be a series of particles of the body in a straight line, which *ipso motûs initio* enjoy absolute rest. This article recals my attention to certain researches in which I had been engaged in connection with the same problem. The conclusions at which I had arrived, although, as far as they went, in harmony with the results of the article to which I have alluded, are not however precisely the same, in consequence of a different definition of the Spontaneous Axis. I had adopted this term to denote the rectilinear locus of a line of particles within the body, all of which, on the application of the impulsive forces, assume a velocity in the direction of the line itself. Thus, according to this definition, a cone struck so as to descend with its axis vertical, whatever be the rotatory motion of the cone, will have its axis of figure for its spontaneous axis. Considering V_x, V_y, V_z constant in equations (6) of Art. III., any two of these equations will represent a straight line. Multiplying them in order by $\frac{L}{A}$, $\frac{M}{B}$, $\frac{N}{C}$, we get, as the condition for their coexistence,

$$\frac{L.V_x}{A} + \frac{M.V_y}{B} + \frac{N.V_z}{C} = \frac{LX}{mA} + \frac{MY}{mB} + \frac{NZ}{mC} \ldots\ldots (\alpha).$$

The direction-cosines of the line are, as appears from the equations to the line, proportional to

$$\frac{L}{A}, \frac{M}{B}, \frac{N}{C}:$$

but, by the definition of the spontaneous axis, these cosines must be also proportional to

$$V_x, V_y, V_z:$$

hence, putting

$$\left.\begin{aligned} V_x &= \frac{kL}{A} \\ V_y &= \frac{kM}{B} \\ V_z &= \frac{kN}{C} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (\beta),$$

we see from (α), that

$$k = \frac{\frac{LX}{mA} + \frac{MY}{mB} + \frac{NZ}{mC}}{\frac{L^2}{A^2} + \frac{M^2}{B^2} + \frac{N^2}{C^2}} \ldots\ldots\ldots (\gamma).$$

From (β) and (γ), V denoting the velocity of the spontaneous axis, we see that

$$\begin{aligned} V &= (V_x^2 + V_y^2 + V_z^2)^{\frac{1}{2}} \\ &= k\left(\frac{L^2}{A^2} + \frac{M^2}{B^2} + \frac{N^2}{C^2}\right)^{\frac{1}{2}} \\ &= \frac{\dfrac{LX}{mA} + \dfrac{MY}{mB} + \dfrac{NZ}{mC}}{\left(\dfrac{L^2}{A^2} + \dfrac{M^2}{B^2} + \dfrac{N^2}{C^2}\right)^{\frac{1}{2}}}\,. \end{aligned}$$

The equations will then become

$$\left.\begin{aligned} \frac{kL}{A} &= \frac{X}{m} + z\,\frac{M}{B} - y\,\frac{N}{C} \\ \frac{kM}{B} &= \frac{Y}{m} + x\,\frac{N}{C} - z\,\frac{L}{A} \\ \frac{kN}{C} &= \frac{Z}{m} + y\,\frac{L}{A} - x\,\frac{M}{B} \end{aligned}\right\} \ldots\ldots\ldots\ldots (\delta),$$

which are the equations to the spontaneous axis, k being supposed to have the value given in the equation (γ).

Suppose that, for the locus of certain points, V_x, V_y, V_z, are constant, without being subject to any other condition except that given by the equations (a). Equations (6) will then represent a straight line parallel to line (δ); and, since V_x, V_y, V_z, may receive an infinite variety of values without violating the essential condition (a), it appears that there is an infinite number of such straight lines in the body.

The locus of those points of which the velocities, irrespectively of their directions, have the same value V, will be a cylinder, of which the equation is

$$\begin{aligned} V^2 = &\left(\frac{X}{m} + z\,\frac{M}{B} - y\,\frac{N}{C}\right)^2 \\ &+ \left(\frac{Y}{m} + x\,\frac{N}{C} - z\,\frac{L}{A}\right)^2 \\ &+ \left(\frac{Z}{m} + y\,\frac{L}{A} - x\,\frac{M}{B}\right)^2, \end{aligned}$$

the number of such surfaces being infinite in accordance with the variations of the value of V and their common axis being the line (δ).

W. W.

VIII.—APPLICATIONS OF THE SYMBOLICAL FORM OF MACLAURIN'S THEOREM.

THE symbolical form of Maclaurin's Theorem may sometimes be applied with advantage to the demonstration of algebraical and trigonometrical formulæ consisting of a finite number of terms. I shall give two instances of its use in such cases. It seems probable that the application of this form of the theorem might be extended to the investigation of the properties of various infinite series in trigonometry: the fundamental principles, however, of its application to such purposes would stand in need of examination.

1. To prove the polynomial theorem.

By Maclaurin's theorem, there is

$$(a + a' + a'' + \ldots)^n = \varepsilon^{a\frac{d}{d0}} . \varepsilon^{a'\frac{d}{d0'}} . \varepsilon^{a''\frac{d}{d0''}} \ldots (0 + 0' + 0'' + \ldots)^n,$$

of which the general term is evidently

$$\frac{a^r . a'^{r'} . a''^{r''} \ldots}{1\ldots r . 1\ldots r' . 1\ldots r'' \ldots} . \left(\frac{d}{d0}\right)^r \left(\frac{d}{d0'}\right)^{r'} \left(\frac{d}{d0''}\right)^{r''} \ldots (0 + 0' + 0'' + \ldots)^n.$$

Now $\left(\frac{d}{d0}\right)^r \left(\frac{d}{d0'}\right)^{r'} \left(\frac{d}{d0''}\right)^{r''} \ldots (0 + 0' + 0'' + \ldots)^n$

$$= \left(\frac{d}{d0}\right)^{r+r'+r''+} \quad 0^n$$

$$= n(n-1)(n-2)\ldots\{n - (r + r' + r'' + \ldots) + 1\}\ 0^{n-r-r'-r''\ldots}$$

which, supposing n to be a positive integer, is always equal to zero, except when

$$n - r - r' - r'' - \ldots = 0,$$

in which case it becomes

$$1.2.3.\ldots n.$$

Hence, if n be a positive integer, the general term of $(a + a' + a'' + \ldots)^n$ is

$$\frac{1.2.3.\ldots n}{1.2\ldots r \times 1.2\ldots r' \times 1.2\ldots r'' \times \ldots}\ a^r a'^{r'} a''^{r''} \ldots$$

under the condition

$$r + r' + r'' + \ldots = n.$$

2. To prove the formula for the development of $(\cos\theta)^n$ by cosines of multiple angles, when n is a positive integer.

Since $2\cos\theta = \cos\theta + \cos(-\theta)$, it is clear that

$$2\cos\theta = (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})\cos 0,$$

$$(2\cos\theta)^2 = (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})\, 2\cos 0\cos\theta$$

$$= (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})\,\{\cos(0+\theta) + \cos(0-\theta)\}$$

$$= (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})(\varepsilon^{\theta\frac{d}{d0'}} + \varepsilon^{-\theta\frac{d}{d0'}})\cos(0+0')$$

$$= (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})(\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})\cos(0+0'),$$

since $\frac{d}{d0'}$, operating on a function of $0+0'$, produces the same result as $\frac{d}{d0}$, $\quad = (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})^2\cos 0.$

Proceeding in the same way, it is obvious that we should get

$$(2\cos\theta)^n = (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})^n\cos 0$$

$$= \Big\{\varepsilon^{n\theta\frac{d}{d0}} + n\,\varepsilon^{(n-2)\theta\frac{d}{d0}} + \frac{n(n-1)}{1.2}\varepsilon^{(n-4)\theta\frac{d}{d0}} + \dots\dots$$
$$+ \frac{n(n-1)}{1.2}\varepsilon^{-(n-4)\theta\frac{d}{d0}} + n\varepsilon^{-(n-2)\theta\frac{d}{d0}} + \varepsilon^{-n\theta\frac{d}{d0}}\Big\}\cos 0$$

$$= \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta + \dots\dots$$
$$+ \frac{n(n-1)}{1.2}\cos\{-(n-4)\theta\} + n\cos\{-(n-2)\theta\} + \cos(-n\theta).$$

3. Putting $\frac{1}{2}\pi - \theta$ for θ in the formula for $(2\cos\theta)^n$, we have

$$(2\sin\theta)^n = \cos\{n(\tfrac{1}{2}\pi-\theta)\} + n\cos\{(n-2)(\tfrac{1}{2}\pi-\theta)\}$$
$$+ \frac{n(n-1)}{1.2}\cos\{(n-4)(\tfrac{1}{2}\pi-\theta)\} + \dots.$$
$$+ \frac{n(n-1)}{1.2}\cos\{-(n-4)(\tfrac{1}{2}\pi-\theta)\}$$
$$+ n\cos\{-(n-2)(\tfrac{1}{2}\pi-\theta)\}$$
$$+ \cos\{-n(\tfrac{1}{2}\pi-\theta)\},$$

a general formula comprehending the four cases ordinarily considered separately, accordingly as n is of the form $4m$, $4m+1$, $4m+2$, or $4m+3$.

W. W.

IX.—ON THE USE OF THE SYMBOL $e^{\theta\sqrt{(-1)}}$ IN CERTAIN TRANSFORMATIONS.

The symbol $e^{\theta\sqrt{(-1)}}$, considered as a sign of affection determining the direction in which a straight line is drawn, may be

N

successfully applied to effect several transformations from rectangular to polar co-ordinates: and the application of the symbol to this purpose will perhaps be useful, not only for the sake of the transformations themselves, but also in the light of illustrating generally the meaning of the symbol $e^{\theta\sqrt{(-1)}}$.

Let it be required to transform the element $dx\,dy$ to polar co-ordinates. We have

$$x = r\cos\theta,$$

$$y = r\sin\theta,$$

$$\therefore\quad x + y\sqrt{(-1)} = r\{\cos\theta + \sqrt{(-1)}\sin\theta\} = re^{\theta\sqrt{(-1)}};$$

differentiating,

$$dx + dy\sqrt{(-1)} = e^{\theta\sqrt{(-1)}}\{dr + rd\theta\sqrt{(-1)}\}\ldots\ldots(1).$$

Now the right hand side of the equation is similar to the left, with the exception of being multiplied by $e^{\theta\sqrt{(-1)}}$, which being only a symbol of direction, need not be considered when the question is merely one of magnitude; therefore, equating possible and impossible parts,

$$\left.\begin{aligned} dx &= dr \\ dy &= rd\theta \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\therefore\quad dx\,dy = rd\theta\,dr,$$

which is the transformation required.

We may also reason thus: the sign of affection $e^{\theta\sqrt{(-1)}}$ merely signifies that the line along which r is measured is inclined at an angle θ to the axis of x; hence, after the differentiation has been performed, we may make $\theta = 0$, or suppose the axis of x to coincide with the direction of r, and thus $dx + dy\sqrt{(-1)} = dr + rd\theta\sqrt{(-1)}$, and equations (2) follow as before.

Again, let it be required to find the effective accelerating forces in the direction of and perpendicular to the radius vector, when the motion is in one plane. We have

$$dx + dy\sqrt{(-1)} = e^{\theta\sqrt{(-1)}}\{dr + rd\theta\sqrt{(-1)}\}$$

$$d^2x + d^2y\sqrt{(-1)} = e^{\theta\sqrt{(-1)}}\{d^2r - rd\theta^2 + \sqrt{(-1)}(rd^2\theta + 2drd\theta)\}\ldots(3);$$

and therefore, from the same reasoning as before,

$$\frac{d^2r}{dt^2} - r\frac{d\theta^2}{dt^2} = \frac{d^2x}{dt^2} = \text{effective accelerating}$$

force in the direction of r,

$$r\frac{d\theta^2}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{d^2y}{dt^2} = \text{effective}$$

accelerating force perpendicular to the direction of r.

The preceding formulæ will enable us to treat very neatly the equations of motion of a disturbed planet in two dimensions. (*Airy's Tracts*, p. 64.)

For we have
$$\left.\begin{aligned}\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}+\frac{dR}{dx}=0\\ \frac{d^2y}{dt^2}+\frac{\mu y}{r^3}+\frac{dR}{dy}=0\end{aligned}\right\}\ldots\ldots\ldots\ldots(4),$$

$$\therefore\ \frac{d^2x+\sqrt{(-1)}\,d^2y}{dt^2}+\frac{\mu}{r^3}\{x+y\sqrt{(-1)}\}+\frac{dR}{dx}+\sqrt{(-1)}\,\frac{dR}{dy}=0,$$

$$\text{or } \frac{d^2r}{dt^2}-r\,\frac{d\theta^2}{dt^2}+\sqrt{(-1)}.\,\frac{1}{r}\,\frac{d}{dt}\left(r^2\,\frac{d\theta}{dt}\right)+\frac{\mu}{r^2}+\frac{dR}{dr}+\sqrt{(-1)}\,\frac{dR}{rd\theta}=0,$$

(where $e^{\theta\sqrt{(-1)}}$ has been put = 1);

$$\left.\begin{aligned}\therefore\ \frac{d^2r}{dt^2}-r\,\frac{d\theta^2}{dt^2}+\frac{\mu}{r^2}+\frac{dR}{dr}=0\\ \frac{d}{dt}\left(r^2\,\frac{d\theta}{dt}\right)+\frac{dR}{d\theta}=0\end{aligned}\right\}\ldots\ldots\ldots(5),$$

which are the equations required.

The same method will apply to other transformations.

H. G.

X.—NOTE ON ORTHOGONAL ISOTHERMAL SURFACES.

In a previous paper in this *Journal* ("On the Equations of the Motion of Heat referred to Curvilinear Co-ordinates," vol. IV. p. 33), I expressed the conditions which must be satisfied by a system of conjugate orthogonal surfaces which are all isothermal, and considered some particular cases of such systems. In addition to those, however, there is another class of surfaces which satisfy the conditions. For it is readily seen that all that is necessary in the demonstration of the theorem relative to cylindrical surfaces in p. 38, is that H_2 shall be independent of λ and λ_1, and that $\dfrac{H_1}{H}$ shall be independent of λ_2. Now a series of concentric spheres (including the case of a series of parallel planes) is such that the value of H_2 at any point of one of them is independent of the position of the point on that surface. Hence we may consider $\lambda_2=a_2$ as representing a series of concentric spheres, and consequently $\lambda=a$, and $\lambda_1=a_1$, a system of conjugate orthogonal cones having the centre of the spheres for their common

vertex. Any cone of one series will intersect any one of the conjugate series along a generating line, and the values of H and H_1 at points along this line will vary as the distances from the centre. Hence $\frac{H_1}{H}$ is independent of λ_2. We therefore see that the demonstration in p. 38 is applicable to the general class of orthogonal cones, as well as to the particular included case of orthogonal cylinders. Thus we have the general theorem that, if a series of cones having a common vertex be isothermal, the series of orthogonal cones will also be isothermal.

This class, and the class of confocal surfaces of the second degree, are all the triple systems which as yet have been found to be isothermal; and, in the paper already referred to, some particular cases were considered in which it was shown that one, or two of the partial series of an orthogonal system are isothermal, and the remaining series not. Since the publication of that paper I have received a Mémoire by M. Lamé, which was published about the same time (*Journal de Mathématiques*, vol. VIII. p. 397, Oct. and Nov. 1843), in which he shows that no other triple isothermal system can exist. This interesting result is the complete answer of a question proposed in this *Journal*, in May 1843 (vol. III. p. 286), and to which a partial answer was given in the paper already referred to (Nov. 1843). The same question has been proposed by M. Bertrand, in the April number of the *Journal de Mathématiques* of the present year (vol. IX. p. 117); and he answers it to a similar extent, by showing that an isothermal series of surfaces has not in every case its two conjugate orthogonal series isothermal also. The reason, however, which he assigns for coming to this conclusion does not seem to be quite satisfactory: for it is founded on the assumption that "it is always possible to take two consecutive isothermal surfaces arbitrarily, and that the law of the temperature of the rest of the body is then determined." Now, by considerations analogous to those brought forward in a paper "On some Points in the Theory of Heat" in this *Journal* (vol. IV. p. 71), it is readily seen that if two consecutive isothermal surfaces be arbitrarily assumed, it will in general be only for *points between them* that a possible system of isothermal surfaces can be determined according to a continuous law. The temperatures of points not lying between them will follow a different law depending on the sources of heat or cold which we must suppose to be distributed over the two assumed surfaces, to retain them at their constant temperatures. Thus, if we

assume arbitrarily two consecutive isothermal surfaces indefinitely near one another, the system of isothermal surfaces through the whole body, to which these two belong, will in general be impossible. In fact, it will generally be impossible to find any two surfaces, containing the two assumed ones between them, which will be such that if they be retained at different constant temperatures, the two assumed surfaces will each be isothermal. But M. Bertrand's conclusion, though correct, is drawn from the assumption that this is generally possible. It may be remarked, however, that though some restriction is necessary in assuming two consecutive surfaces of a possible isothermal system, it will probably be found to be not so narrow as a restriction which M. Bertrand shows to be necessary in choosing two consecutive surfaces of an isothermal series of which the conjugate orthogonal series are isothermal also. If this were previously shown, M. Bertrand's inference would be correct.

M. Bertrand also specially considers the case of isothermal orthogonal surfaces of revolution, and arrives at the interesting theorem that, if each of the conjugate series be isothermal, the traces on the meridian planes will form a system of conjugate isothermal plane curves, or the traces of a system of conjugate isothermal cylinders, on their orthogonal planes.

This follows at once from the equations (14) and (15) (vol. IV. p. 39 of this *Journal*), though it did not occur to me till I saw M. Bertrand's paper. For, from them we deduce

$$\frac{H_1}{H} = \phi(\lambda) \cdot \phi(\lambda_1),$$

which is the sole condition that each of the series of orthogonal plane curves represented by $\lambda = a$, $\lambda_1 = a_1$, shall be isothermal.

Hence we see that if this condition be satisfied, and at the same time the condition expressed by equation (16), the two conjugate orthogonal series of surfaces of revolution will also be isothermal. M. Bertrand states the latter condition in geometrical language as follows—

"Two systems of isothermal orthogonal lines being given, in order that their rotation round an axis may generate isothermal surfaces of revolution, it is necessary that the distances from the four corners of a curvilinear rectangle formed by the given lines shall be the four terms of an analogy."

We may also add, that if a single series of surfaces of revolution be isothermal, and if the traces on a meridian

plane be isothermal lines, then the conjugate orthogonal series of surfaces will also be isothermal.

Also it follows, from the result of M. Lamé's investigations mentioned above, that confocal surfaces of revolution of the second order form the only isothermal system which trace a series of isothermal lines on a meridian plane.

P. Q. R.

XI.—ON THE SOLUTION OF EQUATIONS IN FINITE DIFFERENCES.

By R. L. Ellis, M.A. Fellow of Trinity College.

The partial differential equations which occur in various branches of mathematical physics are, for the most part, of such forms that solutions of them may be obtained without much difficulty. As is well known, the great difficulty in almost all such cases consists in the necessity of determining which of all possible solutions satisfies the particular conditions of the problem on which we are engaged. It seems that before the time of Fourier's researches on heat, the course which mathematicians had uniformly followed was, first to obtain the general solution of the equation of the problem, and then to determine by particular considerations the arbitrary functions which it involved. This course undoubtedly would be the most direct and analytical, were there any general method for determining the form of the functions in question: as, however, there is none, the analytical generality of the first part of the process is in many cases sterile and useless.

Fourier's methods, which depend essentially on the linearity of the partial differential equations which occur in the theory of heat, consist in assuming some simple solution of the equation of the problem, in deducing from hence a more general solution of it, and in determining successively and by means of particular considerations the arbitrary quantities thus introduced in such a manner as to satisfy all the conditions of the question. The general solution with arbitrary functions does not make its appearance in his process; and the reason why it is so much more manageable than the other appears to be, that it is far easier to determine arbitrary constants in accordance with certain conditions than arbitrary functions. There will, generally speaking, be an infinite number of arbitrary constants, and it is therefore necessary to treat them in *classes*. The ingenious synthesis by which this is effected by Fourier, in the different problems discussed by him in the *Théorie de la Chaleur*, forms one of the most interesting parts of that admirable work. The same kind of

reasoning is made use of by Poisson, in his researches on similar subjects: and there can be little doubt that the methods of Fourier, developed and extended as they have been by subsequent writers, will long continue to be an essential element in the application of mathematics to physical researches. Similar methods may be made available in the solution of equations in partial finite differences. Such equations do not, it is true, present themselves very often, as the continuity of the causes to which natural phenomena are due, leads rather to differential equations than to those in finite differences. In fact, I am not aware of any subject, except the theory of probabilities, in which we meet with problems whose solution depends on that of an equation in partial finite differences.

In this theory, however, such problems are not uncommon. One of the most interesting of them, both in its own nature and historically, may serve as an illustration of the application of the methods of Fourier to finite differences. This problem, which has engaged the attention of several writers on the subject of probabilities, and of which a solution was among the earliest efforts of Ampère, is that of the *duration of play*. Professor De Morgan has spoken of this solution and of that of Laplace, as being of the highest order of difficulty: that which I am about to enter on has, I think, a decided advantage in this respect.

The problem itself may be thus stated:—Two persons, M and N, have between them a number a of counters: they play at a game at which M's chance is p, and N's q. The losing player gives one counter to the other, and they are to play on until one or other have lost all his counters. What is the probability that the party will terminate in M's favour after any assigned number of games, N being supposed to have originally x of the a counters?

Let y_{xz} be the probability that M will win the party at the $(z+1)^{\text{th}}$ game. If he win the next game (of which the probability is p), this becomes $y_{x-1\cdot z-1}$; if he lose it (of which the probability is q), it becomes $y_{x+1\cdot z-1}$, and therefore

$$y_{xz} = py_{x-1\cdot z-1} + qy_{x+1\cdot z-1} \dots\dots\dots\dots (1).$$

This is the equation of the problem. It is clear that

$$y_{0z} = 0, \quad y_{az} = 0 \dots\dots\dots\dots (2),$$

as the party ceases as soon as M or N has a counters. Again,

$$y_{x0} = 0 \text{ unless } x = 1, \text{ and } y_{1\cdot 0} = p \dots\dots (3);$$

for if N have more than one counter he cannot lose them all

at the next game; and if he have only one, his chance of his being left without any is p.

Let us assume $\qquad y_{xz} = \alpha^z v_x$ ....(4),

α being arbitrary. Then

$$\alpha v_x = p v_{x-1} + q v_{x+1} \text{ .... (5).}$$

Of this a solution is

$$v_x = C\left(\frac{p}{q}\right)^{\frac{x}{2}} \sin(\mu x + \omega) \text{.......... (6),}$$

where C and ω are arbitrary, and μ such that

$$2\sqrt{(pq)} \cos \mu = \alpha \text{ .... (7):}$$

this form of solution is therefore real if α^2 is less than $4pq$. In order that (4) may satisfy the conditions (2), we must have

$$v_0 = C \sin \omega = 0, \quad v_a = C\left(\frac{p}{q}\right)^{\frac{a}{2}} \sin(\mu a + \omega) = 0 \text{...(8).}$$

It is impossible to satisfy these two conditions without making $C = 0$, which would give a nugatory result, unless $\sin \mu a = 0$ or $\mu = \dfrac{r\pi}{a}$, r being an integer. Let us therefore assume this value for μ; and then, by (7),

$$\alpha = 2\sqrt{(pq)} \cos \frac{r\pi}{a} \text{ .... (9).}$$

In order to satisfy (8), we have now only to make $\omega = 0$, and then, substituting the values of v_x and α in (4), we get

$$y_{xz} = (4pq)^{\frac{z}{2}} \left(\frac{p}{q}\right)^{\frac{x}{2}} C \sin \frac{r\pi}{a} x \left(\cos \frac{r\pi}{a}\right)^z \text{....(10).}$$

This value, in which C and r are arbitrary, satisfies (1) and (2), and in consequence of the linearity of these equations they will be satisfied by a sum of similar values, and we shall thus have a more general solution, viz.

$$y_{xz} = (4pq)^{\frac{z}{2}} \left(\frac{p}{q}\right)^{\frac{x}{2}} \Sigma C \sin \frac{r\pi}{a} x \left(\cos \frac{r\pi}{a}\right)^z \text{....(11).}$$

If in this we put $z = 0$, we have

$$y_{x0} = \left(\frac{p}{q}\right)^{\frac{x}{2}} \Sigma C \sin \frac{r\pi}{a} x \text{ .... (12).}$$

Now, by (3), this is to be equal to p for $x = 1$, and to 0 for the $a - 2$ values of x, $2 . 3. \ldots a - 1$. There are thus $a - 1$ conditions for (12) to fulfil, and therefore we have, extending the summation Σ from $r = 1$ to $r = a - 1$, the following system of equations:

$$\left.\begin{aligned}\sqrt{(pq)} &= C_1 \sin \frac{\pi}{a} + \ldots + C_{a-1} \sin \frac{a-1}{a}\pi \\ 0 &= C_1 \sin \frac{2\pi}{a} + \ldots + C_{a-1} \sin 2\frac{a-1}{a}\pi \\ 0 &= \ldots\ldots\ldots\ldots \\ 0 &= C_1 \sin \frac{a-1}{a}\pi + \ldots + C_{a-1} \sin \frac{(a-1)^2}{a}\pi\end{aligned}\right\} \ldots\ldots (13).$$

From these $a-1$ equations we have to determine the $a-1$ quantities $C_1 \ldots C_{a-1}$. In order to do this, multiply the first equation by $\sin \frac{r}{a}\pi$, the second by $\sin 2\frac{r}{a}\pi$, and so on, (r being an integer less than a), and add. Then, as may be easily shown, the coefficient of every one of the quantities C, except C_r, will in the resulting sum be equal to zero, while that of C_r will be $\frac{a}{2}$. Consequently (13) is equivalent to the system of equations included in the general formula

$$C_r = \frac{2}{a}\sqrt{(pq)} \sin \frac{r}{a}\pi \ldots\ldots\ldots\ldots (14),$$

and consequently (11) becomes

$$y_{xz} = \frac{1}{a}(4pq)^{\frac{z+1}{2}} \left(\frac{p}{q}\right)^{\frac{x}{2}} \Sigma_1^{a-1} \sin \frac{r}{a}\pi \sin \frac{rx}{a}\pi \left(\cos \frac{r}{a}\pi\right)^z \ldots (15),$$

which is the required probability.

We may deduce from this formula by indirect considerations, one or two analytical theorems. For it is obviously impossible that the party should terminate in *M*'s favour in less than x games, as x is the number of counters he must win from *N*. Consequently

$$\Sigma_1^{a-1} \sin \frac{r}{a}\pi \sin \frac{rx}{a}\pi \left(\cos \frac{r}{a}\pi\right)^z = 0 \ldots\ldots (16)$$

for all integer values of z less than $x-1$.

Again, *M* may win the party at the x^{th} game, if he win x games in succession, the probability of which is p^x. Hence, putting $z = x-1$, we have

$$p^x = \frac{1}{a}(4pq)^{\frac{x}{2}} \left(\frac{p}{q}\right)^{\frac{x}{2}} \Sigma_1^{a-1} \sin \frac{r}{a}\pi \sin \frac{rx}{a}\pi \left(\cos \frac{r}{a}\pi\right)^{x-1},$$

$$\text{or} \quad \Sigma_1^{a-1} \sin \frac{r}{a}\pi \sin \frac{rx}{a}\pi \left(\cos \frac{r}{a}\pi\right)^{x-1} = \frac{a}{2^x} \ldots\ldots (17).$$

These formulæ may undoubtedly be established by other

methods, but I have thought it worth while to point out this way of deducing them, from the analogy it bears to that in which many remarkable theorems are obtained by Poisson, in his *Théorie de la Chaleur,* namely by considering the nature of the quantities which his formulæ represent. This mode of establishing analytical theorems by considerations founded on the interpretation of our results, is one of the most curious features of the more recent methods of treating physical questions.

To (16) and (17) another theorem may be added, by the following consideration. M, if he win, must win the party either in x games or in x + an even number of games. For if he lose k games he must win back k games and x more or there must have been $x + 2k$ games in the party. Hence his chance is zero whenever $z + 1 = x + 2k + 1$, and therefore

$$\Sigma_1^{a-1} \sin \frac{r}{a}\pi \sin \frac{rx}{a}\pi \left(\cos \frac{r}{a}\pi\right)^{x+2k} = 0 \ldots . (18),$$

k being any positive integer whatever.

When a is infinite, the sums contained in the three last equations become definite integrals. Let

$$\frac{r}{a}\pi = \phi, \quad \text{then} \quad \frac{\pi}{a} = d\phi, \quad \text{and} \quad \frac{a-1}{a}\pi = \pi.$$

Consequently (16), (17), (18), become respectively

$$\int_0^\pi \sin \phi \sin x\phi (\cos \phi)^z d\phi = 0 \ldots\ldots .(19),$$

(z being integral and less than $x - 1$),

$$\int_0^\pi \sin \phi \sin x\phi (\cos \phi)^{x-1} d\phi = \frac{\pi}{2^x} \ldots\ldots (20),$$

$$\int_0^\pi \sin \phi \sin x\phi (\cos \phi)^{x+2k} d\phi = 0 \ldots\ldots (21).$$

If, instead of seeking the probability that M will win the party *at* the $(z + 1)^{\text{th}}$ game, we wished to find that of his winning it *after* z or more games shall have been played, we should only have to sum (15) for z from z to infinity. Calling this new probability u_{xz}, we should thus get

$$u_{xz} = \frac{1}{a}(4pq)^{\frac{z+1}{2}}\left(\frac{p}{q}\right)^{\frac{x}{2}} \Sigma_1^{a-1} \frac{\sin \frac{r}{a}\pi \sin \frac{rx}{a}\pi}{1-2\surd(pq)\cos \frac{r}{a}\pi}\left(\cos \frac{r}{a}\pi\right)^z \ldots (22).$$

If in (22) we put z equal to zero, we have then the probability of M's winning the party at the first, second, &c.

games, *i.e.* of his winning it at all. Writing simply u_x for u_{x0}, we shall thus get

$$u_x = \frac{2}{a}\sqrt{(pq)}\left(\frac{p}{q}\right)^{\frac{x}{2}} \Sigma_1^{a-1} \frac{\sin\frac{r}{a}\pi \sin\frac{rx}{a}\pi}{1 - 2\sqrt{(pq)}\cos\frac{r}{a}\pi} \dots\text{(23)}.$$

Now of this probability we can obtain, as is well known, a much simpler expression. For it is easily seen that we shall have

$$u_x = pu_{x-1} + qu_{x+1} \dots\dots\dots\text{(24)}$$

for every value of x, provided that, instead of considering u_x as the probability that M *will* win the party, we make it denote the probability that he either *has* won or *will* win it. As it is impossible that he can have already won it while x differs from zero, this alteration does not affect the value represented by u_x except for the case of $x = 0$. In this case the value of u_x, as expressed by (23), will be zero, as the party is at an end, M having already won it. But according to the proposed modification, the new value of u_0 will be unity, and therefore we have for the initial and final values of u_x,

$$u_0 = 1, \quad u_a = 0 \dots\dots\dots\text{(25)}.$$

The necessity of this modification arises from this, that otherwise the relation expressed by (24) would not be in all cases true. For when $x = 1$, we should have $u_1 = qu_2$, whereas the true value is of course $u_1 = p + qu_2$.

From (24) we have (introducing the relation $p + q = 1$)

$$u_x = \alpha + \beta\left(\frac{p}{q}\right)^x \dots\dots\dots\text{(26)},$$

α and β being arbitrary constants: and thence, by (25), we get

$$1 = \alpha + \beta,$$

$$0 = \alpha + \beta\left(\frac{p}{q}\right)^a,$$

and consequently $$u_x = \frac{\left(\frac{p}{q}\right)^x - \left(\frac{p}{q}\right)^a}{1 - \left(\frac{p}{q}\right)^a} \dots\dots\dots\text{(27)}.$$

This expression is therefore, except for $x = 0$, equivalent to (23), into which however the relation already mentioned, viz. that $p + q = 1$ has not as yet been introduced.

When p and q are equal, (27) becomes

$$u_x = \frac{a-x}{a} \quad \ldots\ldots\ldots\ldots\ldots \text{(28)},$$

while (23) similarly becomes

$$u_x = \frac{1}{a}\,\Sigma_1^{a-1}\,\frac{\sin\frac{r}{a}\pi\,\sin\frac{rx}{a}\pi}{1-\cos\frac{r}{a}\pi},$$

$$\text{or}\quad u_x = \frac{1}{a}\,\Sigma_1^{a-1}\cot\frac{r}{a}\frac{\pi}{2}\sin\frac{rx}{a}\pi \ldots\ldots\ldots \text{(29)}.$$

Comparing (28) and (29), we have the following theorem: writing x for $a-x$,

$$x = \Sigma_1^{a-1} \pm \cot\frac{r}{a}\frac{\pi}{2}\sin\frac{rx}{a}\pi \ldots\ldots\ldots \text{(30)},$$

the upper sign to be taken when r is odd.

This theorem, like the preceding ones (16), (17), &c., requires x not to transgress the limits $x = 1$, $x = a - 1$. In the case supposed (viz. when p and q are each equal to $\frac{1}{2}$), (22) becomes

$$u_{xz} = \frac{1}{a}\,\Sigma_1^{a-1}\cot\frac{r}{a}\frac{\pi}{2}\sin\frac{rx}{a}\pi\left(\cos\frac{r}{a}\pi\right)^z \ldots \text{(31)}.$$

But as the party cannot be won in less than x games, $u_{x0} = u_{xz}$ while z is less than x, and therefore

$$x = \Sigma_1^{a-1} \pm \cot\frac{r}{a}\frac{\pi}{2}\sin\frac{rx}{a}\pi\left(\cos\frac{r}{a}\pi\right)^z \ldots \text{(32)},$$

of which (30) is a particular case.

If, instead of seeking the probability that at the $(z+1)^{\text{th}}$ game N would lose the party, by losing the last of his x counters, we had sought that of his having at the termination of this game any assigned number of counters k, the following method might have been made use of.

Let y_{xz} be the probability in question. It is clear that it will satisfy, as before, equations (1) and (2). But instead of (3), we shall in this case have

$$y_{x\cdot 0} = 0,\ \text{unless}\ x = k \pm 1,\ \text{and}\ y_{k\pm1\cdot 0} = \tfrac{1}{2}\{p + q \pm (p - q)\} \ldots \text{(3)}.$$

Equation (11) therefore, which depends merely on (1) and (2), will still obtain; but instead of the system of equations (13), we shall have the following:

$$\left.\begin{aligned}
0 &= C_1 \sin\frac{\pi}{a} + \ldots + C_{a-1}\sin\frac{a-1}{a}\pi \\
\&c. &= \&c. \\
\left(\frac{q^{k+1}}{p^{k-1}}\right)^{\frac{1}{2}} &= C_1 \sin\frac{k-1}{a}\pi + \ldots + C_{a-1}\sin\frac{(k-1)(a-1)}{a}\pi \\
0 &= \&c. \\
\left(\frac{q^{k+1}}{p^{k-1}}\right)^{\frac{1}{2}} &= C_1 \sin\frac{k+1}{a}\pi + \ldots + C_{a-1}\sin\frac{(k+1)(a-1)}{a}\pi \\
\&c. &= \&c. \\
0 &= C_1 \sin\frac{a-1}{a}\pi + \ldots + C_{a-1}\sin\frac{(a-1)^2}{a}\pi
\end{aligned}\right\}\ldots(13').$$

From whence, by the same system of factors as before, we deduce the general formula

$$C_r = \frac{4}{a}\left(\frac{q^{k+1}}{p^{k-1}}\right)^{\frac{1}{2}} \sin\frac{kr}{a}\pi \cos\frac{r}{a}\pi \ldots\ldots (14');$$

for the factors corresponding to the two equations whose first members are different from zero, are $\sin\frac{(k-1)r}{a}\pi$ and $\sin\frac{(k+1)r}{a}\pi$, and the sum of these is $2\sin\frac{kr}{a}\pi\cos\frac{r}{a}\pi$. Consequently the expression of the probability sought will be (accenting the y for distinctness),

$$y'_{xz} = \frac{2}{a}(4pq)^{\frac{z+1}{2}}\left(\frac{p}{q}\right)^{\frac{x-k}{2}} \Sigma_1^{a-1} \sin\frac{kr}{a}\pi \sin\frac{rx}{a}\pi\left(\cos\frac{r}{a}\pi\right)^{z+1} \ldots(15').$$

It is an obvious consequence of the discontinuity of the limiting conditions of the problem, that this expression does not reduce itself to (15) when k is taken equal to zero. For the same reason it is not applicable when k is equal to unity: and on the other hand, it is not to be greater than $a-2$.

It is unnecessary to trace the different corollaries deducible from the last written equation, as it has been introduced merely to illustrate the facility with which our method discusses any proposed modification of the question of the duration of play.

One point, which is perhaps worth notice, is the symmetrical manner in which x and p, k and q, enter into (15'): the result, however, which is the interpretation of this symmetry may probably be obtained by general considerations.

A more general question would arise from supposing it possible for M to win or lose at each game any number of counters not greater than a. The method we have been

illustrating would apply to this question, but the solution of it involves that of an algebraical equation of a degree superior to the second.

Another part of the subject, namely, the numerical calculation of the expressions already obtained, would not be consistent with the design of this paper. When z is sufficiently large, all the summations with respect to r may be reduced to their first and last terms, unless a is extremely large, in which case other methods of approximating (those, namely, of Laplace), may be made use of.

Enough has probably been said to show the facility which the method I have proposed is capable of giving to questions of acknowledged difficulty. I am not aware that it has been before pointed out; but as I am not at present able to refer to any work on the subject, I cannot speak confidently on this point. [x, a are integral throughout.]

XII.—NOTE ON GEOMETRICAL DISCONTINUITY.

If parallel straight lines be drawn cutting an ellipse, and from the points of section normals be drawn, the intersections of the pairs of normals will all lie on an hyperbola concentric with the ellipse.

This problem presents a somewhat singular instance of geometrical discontinuity: the equation to the hyperbola will be found to be

$$(a^2x \sin \alpha - b^2y \cos \alpha)(x \cos \alpha - y \sin \alpha)$$
$$= \sin \alpha \cos \alpha (a^2 - b^2)^2 \left(\frac{a^2 \sin^2 \alpha - b^2 \cos^2 \alpha}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}\right)^2 \ldots (A),$$

where α is the angle at which the straight lines cut the axis major.

Now the above equation is found by considering the problem as that of finding the locus of the intersections of pairs of lines drawn according to an assigned law, and therefore we might say that the equation represented the locus of the intersections of the normals: this, however, would not be strictly correct, for the locus of the intersections is not an hyperbola, but only a small arc of an hyperbola, as may be seen without much difficulty. If the lines be drawn parallel to either of the axes, the hyperbola degenerates into two straight lines, but the intersections of the normals only occupy a finite portion of these lines, those portions, in fact, which lie between the centres of curvature at the extremities

of the axes, or between the cusps of the evolute; and more generally the portion of the hyperbola whose equation is (A), belonging to the problem, is the arc lying within the evolute of the ellipse. In the case of a circle $a = b$, and the hyperbola again degenerates into two straight lines, but the only portion of them belonging to the problem is their point of intersection. H. G.

XIII.—NOTE ON THE LAW OF GRAVITY AT THE SURFACE OF A REVOLVING HOMOGENEOUS FLUID.

It has been shown by Maclaurin that a homogeneous fluid, revolving uniformly round a fixed axis, and acted upon only by the attractive force of its own particles, may, with the same angular velocity, have two different figures of equilibrium, each a spheroid of revolution round the shorter axis: and Jacobi has shown that it may be in equilibrium in the form of an ellipsoid with three unequal axes, the shortest coinciding with the axis of rotation. The following simple consideration determines the law of gravity at the surface in each case.

Let any surface concentric with the free surface of the fluid, and similar to it, be described in the interior of the fluid. If all the fluid exterior to the surface were removed, the fluid would still be in equilibrium, since the *proportions* of the free surface depend only on the density and angular velocity. Hence the accelerating force at this surface, as far as it is due to the centrifugal force, and the attraction of the interior mass, must be everywhere perpendicular to the surface. But the mass without it, being contained between two concentric similar ellipsoids exerts no attraction on any point in the surface, and therefore the direction of the accelerating force on any point of this surface in the interior of the fluid is in the normal. Hence the surface must be of equal pressure. Now, let it be supposed to approach the free surface so as to be indefinitely near it. In order that the pressure on every point of it may be the same, the accelerating force on any point of the indefinitely thin shell between it and the free surface, or the force of gravity at any point of the free surface, must be inversely proportional to the thickness of the shell at the point, or inversely as the perpendicular from the centre to the tangent plane at the point. A very simple analytical proof of this result was given by Liouville (*Journal de Mathématiques*, vol. VIII. p. 360). We may also state it, that the force of gravity at any point of the free surface is inversely proportional to

the electrical tension at the point, supposing the surface an electrified conductor. P. Q. R.

XIV.—MATHEMATICAL NOTE.

VARIOUS solutions have appeared from time to time of the following problem: it is hoped that the one here given may claim attention from its simplicity.

"The circle which passes through the intersections of three tangents to a parabola, passes also through the focus."

Let

$$y = x \tan \theta_1 + m \cot \theta_1 \ldots\ldots\ldots\ldots (1),$$

$$y = x \tan \theta_2 + m \cot \theta_2 \ldots\ldots\ldots\ldots (2),$$

$$y = x \tan \theta_3 + m \cot \theta_3 \ldots\ldots\ldots\ldots (3),$$

be the equations to three tangents to a parabola making angles $\theta_1\ \theta_2\ \theta_3$ with the axis of x. Let x', y', be the intersection of (1) and (2); then we easily find

$$x' = \frac{m \cos \theta_1 \cos \theta_2}{\sin \theta_1 \sin \theta_2}, \qquad x' - m = m\, \frac{\cos(\theta_1 + \theta_2)}{\sin \theta_1 \sin \theta_2},$$

$$y' = m\, \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1 \sin \theta_2}.$$

Therefore

$$(x' - m)^2 + y'^2 = m^2 \left\{ \frac{\cos^2(\theta_1 + \theta_2)}{\sin^2 \theta_1 \sin^2 \theta_2} + \frac{\sin^2(\theta_1 + \theta_2)}{\sin^2 \theta_1 \ \sin^2 \theta_2} \right\}$$

$$= \frac{m}{\sin \theta_1 \sin \theta_2} \cdot \frac{m \sin \theta_3}{\sin \theta_1 \sin \theta_2 \sin \theta_3}$$

$$= \frac{m}{\sin \theta_1 \sin \theta_2} \cdot \frac{m \sin \{(\theta_1 + \theta_2 + \theta_3) - (\theta_1 + \theta_2)\}}{\sin \theta_1 . \sin \theta_2 \sin \theta_3}$$

$$= \frac{m \sin(\theta_1 + \theta_2 + \theta_3)}{\sin \theta_1 \sin \theta_2 \sin \theta_2} \cdot \frac{m \cos(\theta_1 + \theta_2)}{\sin \theta_1 \sin \theta_2}$$

$$- \frac{m \cos(\theta_1 + \theta_2 + \theta_3)}{\sin \theta_1 \sin \theta_2 \sin \theta_3} \cdot \frac{m \sin(\theta_1 + \theta_2)}{\sin \theta_1 \sin \theta_2}$$

$$= \frac{m \sin(\theta_1 + \theta_2 + \theta_3)}{\sin \theta_1 \sin \theta_2 \sin \theta_3} . (x' - m) - \frac{m \cos(\theta_1 + \theta_2 + \theta_3)}{\sin \theta_1 \sin \theta_2 \sin \theta_3} . y' \ldots (4).$$

As this relation is symmetrical with regard to $\theta_1\ \theta_2\ \theta_3$, it will hold at the other two intersections. Hence the circle of which the equation is (4) passes through the intersections of the three tangents of which the equations are (1), (2), (3); but (4) is evidently satisfied by $x = m$, $y = 0$, and therefore the circle passes through the focus.

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